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Hydrodynamics. — *On the distinction between irregular and systematic motion in diffusion problems.* By J. M. BURGERS. (Mededeeling No. 41 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of March 29, 1941.)

1. The term "diffusion" is used to denote the gradual spreading of some form of matter through a medium of different nature, from regions where this matter is present in high concentration to regions from where it is absent, or where the concentration is low. The diffusion is the consequence of irregular movements to which the particles of the diffusing matter are subjected, and which may find their origin either in the molecular motion of the medium in which the particles are dispersed, or in coarser, turbulent motions present in this medium. The "irregularity" of the movements of the diffusing particles expresses itself in the circumstance that these movements do not show any preference for a particular direction: a given particle has the same chance to be driven in a positive as in a negative direction. When this equality of chances is not found, we say that along with the "irregular" movements there is present a certain "systematic" motion, the cause of which perhaps may be looked for in the action of forces which tend to drive the particles in a definite direction, or in the presence of a systematic flow in the medium in which the particles are dispersed.

The fact that the "irregular" movements to which the particles are subjected, notwithstanding their indifference as regards direction, can bring about a gradual transfer of matter from regions of high concentration to such of low concentration, is a statistical effect; it is a consequence of the circumstance that many more particles are set into motion from regions of high concentration than from other regions. It will be evident that in this process the intensity of the "irregular" movements likewise plays a part: when in a certain region this intensity is much higher than elsewhere, this circumstance in itself already will bring about that the diffusion from the said region will be more important than the diffusion towards it.

2. In order to translate these considerations into formulae we introduce the components u, v, w of the velocities of the particles. When the movements are exclusively of "irregular" character, the mean values of these components, taken over a suitable interval of time, should be zero:

$$\bar{u} = \bar{v} = \bar{w} = 0 \quad \ldots \ldots \ldots \ldots \quad (1)$$

The intensity of the irregular movements can be defined by giving the mean values of the squares of these components: $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, or the mean value of the square of the resulting velocity: $\overline{u^2 + v^2 + w^2}$. These quantities either may be constant throughout the whole field, or they may change from point to point. — When at every point of the field

$$\overline{u^2} = \overline{v^2} = \overline{w^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the irregular motion is said to be isotropic; when these quantities differ from each other it is anisotropic.

As mentioned above the "irregular" motion may be accompanied by a systematic one, and a great number of problems refer to such cases. An important example is the combination of diffusion with a systematic downward movement due to the action of gravity on the particles; a common problem then is to find the concentration gradient which will balance the action of gravity and which thus can be stationary.

Clearly the proper distinction between "irregular" and "systematic" motions will play an important part in the analysis of such problems, for only when the "systematic" motion has been taken apart it will become possible to find the connection between diffusion, intensity of the "irregular" movements, and concentration gradient. It is the object of the present note to give some attention to this matter.

3. It will be expected that a "systematic" motion can be detected by taking the *mean values* of the velocity components u, v, w, as the "irregular" parts of these quantities then will be eliminated. In what way, however, these "mean values" should be defined?

In order to fix ideas we start from the assumption that the precise movement of every particle shall be known as a function of the time; thus *e.g.*, for the particle numbered i, the functions $u_i(t)$, $v_i(t)$, $w_i(t)$ will be considered as given. We then may take mean values of these quantities over a certain interval of time, say from $t = t_0$ until $t = t_0 + T$, and define the systematic part of the motion of the particle by the formula:

$$u_{is} = \frac{1}{T} \int_{t_0}^{t_0+T} u_i \, dt, \quad \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be evident that in such a mean value an amount of uncertainty, of unsharpness, must be taken for granted: the interval T must be of sufficient length compared with the "periods" of the irregular movements; on the other hand, when it is taken too long, the particle in the meantime may have wandered from one region of the field towards another, where the state of motion may be different. Some intermediate way must be found between these two extremes; where this is impossible a proper distinction between "systematic" and "irregular" motions cannot be arrived at.

It will be expected that when the same calculation is performed for various particles which at the moment t_0 were lying in the same region of the field, the result of formula (3) must be practically the same for all of them. (For simplicity it will be assumed that all particles are of the same kind.) It is possible therefore to obtain a more precise value of the "systematic" motion by taking the average of the quantity defined by (3) over a great number N of such particles:

$$u_s = \frac{1}{N} \sum_{i=1}^{i=N} u_{is} = \frac{1}{NT} \sum_{i=1}^{i=N} \int_{t_0}^{t_0+T} u_i \, dt \quad \ldots \ldots \quad (4)$$

where it is understood that the summation refers to the particles which at $t = t_0$ were lying in the same region, *i.e.* in the same element of volume ω of the field.

4. We may expect that when N is sufficiently large it will not be necessary to make the interval T very long. Hence the uncertainty present in the definition of the systematic motion when considered as a function of the time, to which reference was made above, can be reduced by considering many particles; however, in those cases where the distances between the individual particles are great, this will require the consideration of a rather extensive region and consequently may introduce an amount of unsharpness in the notion of the systematic motion as a function of the spatial coordinates.

The interval T nevertheless must not be reduced below a certain limit: it must be sufficiently ample in order that the motion of practically all particles considered will have suffered one or more "irregular" disturbances during this interval.

Indeed it will be evident that when T is diminished without limit in formula (4), we arrive at a quantity:

$$\bar{u} = \frac{1}{N} \sum_{i=1}^{i=N} u_i \, (t_0) \quad \ldots \ldots \ldots \quad (5)$$

which is *not* identical with u_s. The quantity $\bar{u}$ defined by (5) is the mean value of the velocities at the instant t_0 of the particles which at that instant are lying in a given element of volume ω; in this quantity "irregular" diffusion and "systematic" flow are combined in such a way that they cannot be distinguished from each other. This can be seen for instance by imagining a case of pure diffusion, from a region A of high concentration towards a region B of low concentration, without any systematic action either of external forces or of flow of the medium in which the particles are dispersed. When at a given instant we consider the particles lying in an element of volume situated between these two regions, the chance of finding particles coming from the region A will be greater than the chance of finding particles from the region B; amongst the particles first mentioned,

however, velocities in the direction from A to B will be preponderant, whereas the other type of particles will show a preponderance of velocities in the opposite direction. Hence the value of $\bar{u}$ given by (5) will not be zero, although we have supposed that there is no systematic motion.

When the mean velocity $\bar{u}$, and the other two components $\bar{v}$ and $\bar{w}$ defined by two similar equations, are multiplied by the average number of particles n per unit volume, to be found in the element of volume ω at $t = t_0$, we obtain quantities:

$$q_x = n\bar{u}, \qquad q_y = n\bar{v}, \qquad q_z = n\bar{w}. \quad \ldots \ldots \quad (6)$$

which evidently will represent the components of the "resulting mean current of particles" (measured as the number of particles which in unit time cross an element of surface of unit area). The components q_x, q_y, q_z will satisfy the equation of continuity:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = - \frac{\partial n}{\partial t} \quad \ldots \ldots \quad (7)$$

In the particular case of stationary motion in the direction of the coordinate x only, this equation gives: $q_x = constant$. When the field is limited by impermeable boundaries, the current must be zero at these boundaries; hence we obtain: $q_x = 0$ everywhere, from which it follows that in this case: $\bar{u} = 0$. Such a case can be found when both pure diffusion and systematic motion are present, of such intensities that they balance each other.

5. Returning to the "systematic" motion as defined by eq. (4), it is possible to write down the following formula:

$$u_s = \frac{1}{N} \sum_{i=1}^{i=N} u_i(t). \quad \ldots \ldots \quad (8)$$

where $t = t_0 + T$. This formula must give a good approximation to u_s, provided the interval T is not too short. It expresses the property that whereas the mean value of the velocities at the instant t_0 of the particles which at that instant are lying in a given element ω, will define the "mean resulting velocity of transportation" of the particles (related to the "mean resulting current"), on the other hand the mean value of the velocities at the instant $t_0 + T$ of the same particles will give the "systematic" velocity, provided the interval T is sufficiently long in order that practically all particles may have suffered one or more irregular disturbances during this interval.

6. The results obtained can be expressed in a different form, in which it is not necessary to know the functions $u_i(t)$, etc. for each particle explicitly, and which is more adapted to statistical considerations. Restrict-

ing for simplicity to motion in the direction of the coordinate x only, we introduce a frequency function:

$$f(\xi, l, u)\, d\xi\, dl\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which shall give the number of particles satisfying the following conditions:

(1) at the time t_0 the particles are located in the element ω, *i.e.* between ξ and $\xi + d\xi$;

(2) in the interval from t_0 until $t = t_0 + T$ the particles have moved over distances lying between l and $l + dl$;

(3) at the time $t = t_0 + T$ the velocities of the particles have values between u and $u + du$.

Theoretically, for the case of an unlimited field, we may assume that l and u may run from $-\infty$ to $+\infty$, although values exceeding certain limits practically never may occur.

Along with f we introduce a second function F defined by:

$$F(x, l, u) = f(x - l, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then $F(x, l, u)\, dx\, dl\, du$ represents the number of particles satisfying the conditions:

(1) at the time t the particles are located in a certain element Ω extending from x towards $x + dx$;

(2) in the interval from $t_0 = t - T$ until t the particles had moved over distances lying between l and $l + dl$;

(3) at the time t the velocities of the particles have values between u and $u + du$.

Evidently we have:

$$f(x, l, u) = F(x + l, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

With the aid of these functions the quantities considered above can be given in the form of integrals as follows (for shortness the limits $-\infty$, $+\infty$ have not been written out):

(I) The number of particles per unit of volume in the element ω at the time t_0 is given by:

$$n(\xi, t_0) = \int dl \int du\, f(\xi, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The same for the element Ω at t_0:

$$n(x, t_0) = \int dl \int du\, f(x, l, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

(II) The number of particles per unit of volume in the element Ω at t is given by:

$$n(x, t) = \int dl \int du\, F(x, l, u) = \int dl \int du\, f(x - l, l, u) \quad . \quad . \quad . \quad (12)$$

(III) The "mean resulting current" of particles through Ω at t is given by:

$$q(x, t) = \int dl \int du\, u\, F(x, l, u) = \int dl \int du\, u\, f(x - l, l, u) \quad . \quad (13)$$

From this quantity the "mean resulting velocity of transportation" of the particles in Ω at t is obtained by division through $n(x, t)$:

$$\bar{u}(x, t) = \frac{q(x, t)}{n(x, t)} = \frac{1}{n(x, t)} \int dl \int du\, u\, F(x, l, u) \quad . \quad . \quad . \quad (13a)$$

(IV) The "systematic velocity", defined at the time t, of the particles which at t_0 were located in the element ω, is obtained from:

$$(u_s)_\omega = \frac{1}{n(\xi, t_0)} \int dl \int du\, u\, f(\xi, l, u) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The same for the particles which at t_0 were located in Ω:

$$(u_s)_\Omega = \frac{1}{n(x, t_0)} \int dl \int du\, u\, f(x, l, u) \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

7. In order now to arrive at the distinction between "systematic motion" and "diffusion", and at the same time to obtain the connection between the latter and the intensity of the "irregular" movements, etc., we make use of a procedure which is analogous to one applied by FOKKER and by PLANCK in certain statistical considerations [1]). We introduce the hypothesis that the function $f(x-l, l, u)$ can be developed as follows:

$$f(x-l, l, u) = f(x, l, u) - l\,\frac{\partial f(x, l, u)}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Formula (13) can then be transformed into:

$$\left.\begin{aligned} q(x, t) &= \int dl \int du\, u\, f(x, l, u) - \int dl \int du\, u\, l\, \frac{\partial f(x, l, u)}{\partial x} = \\ &= n(x, t_0) \,.\, (u_s)_\Omega - \frac{\partial}{\partial x} \int dl \int du\, u\, l\, f(x, l, u) \end{aligned}\right\} \quad . \quad . \quad (16)$$

We shall write:

$$\overline{u\,l} = \frac{1}{n(x, t_0)} \int dl \int du\, u\, l\, f(x, l, u) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that $\overline{u\,l}$ will be the mean value of the product $u\,l$ at the time t, calculated for the particles which at t_0 were located in Ω. With a simplification of notation (16) consequently may be written:

$$q = n_0 \,.\, u_s - \frac{\partial}{\partial x}\,(n_0 \,.\, \overline{u\,l}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

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1) A. D. FOKKER, Ann. d. Physik (IV) 43, 812 (1914); and in particular: Sur les mouvements Browniens dans le champ du rayonnement noir, Archives Néerlandaises des Sciences exactes etc. (III A) 4, 379 (1918).

M. PLANCK, Ueber einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie, Sitz. Ber. Berliner Akademie, 324 (1917).

In this equation a separation has been obtained between "systematic motion" and "diffusion". It will be seen that the intensity of the diffusion is dependent upon the magnitude of the quantity $\overline{u\,l}$, which is defined by (17). Now this is a quantity related to those which occur in the "mixture length theories" of transfer of momentum and diffusion in turbulent fluid motion. It is very gratifying that in eq. (18) it makes its appearance in a direct way, and moreover in the more exact form in which it has been defined by TAYLOR [2]).

8. It must be observed, however, that although the current q in eq. (18) refers to the instant t, the number of particles n_0 occurring in this equation refers to the instant $t_0 = t - T$. Now it has been remarked already before that the distinction between "systematic" and "irregular" motion properly can be made only when the interval T may be chosen so that during this interval the particles have not moved too far from their original positions; nor should the state of the field have changed appreciably in it. When this is the case, we may assume:

$$n\,(x,\,t) \simeq n\,(x,\,t_0),$$

so that eq. (18) will become:

$$q \simeq n \,.\, u_s - \frac{\partial}{\partial x}(n \,.\, \overline{u\,l}) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

In this form the equation has obtained a more convenient appearance. The degree of approximation involved in it is a necessary consequence of the unsharpness which is a priori connected with the definition of "systematic motion", as was shown before (see 3).

Nevertheless it may be useful to transform eq. (18) in a different way, which moreover makes it possible to eliminate from the second term a residue of "systematic motion", which still may be present in it.

For this purpose from eq. (11a) we deduce:

$$\begin{aligned} n\,(x,\,t_0) &= \int dl \int du\, F\,(x+l,\,l,\,u) = \\ &= \int dl \int du\, F(x,\,l,\,u) + \int dl \int du\; l\, \frac{\partial\, F(x,\,l,\,u)}{\partial x} = \\ &= n\,(x,\,t) + \frac{\partial}{dx}(n\,\bar{l}) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19) \end{aligned}$$

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[2]) G. I. TAYLOR, Diffusion by continuous movements, Proc. London Math. Society (2) **20**, 196 (1922); Statistical theory of turbulence, Proc. Roy. Soc. London **A 151**, pp. 423 *seqq.* See also: S. GOLDSTEIN, Modern developments in fluid dynamics (Oxford 1938) I, pp. 205, 217.

In a similar way (17) can be transformed into

$$n(x, t_0) \cdot \overline{ul} = \int dl \int du\, ul\, F(x+l, l, u) =$$

$$= \int dl \int du\, ul\, F(x, l, u) + \int dl \int du\, ul^2 \frac{\partial F(x, l, u)}{dx} =$$

$$= n(x, t) \cdot \overline{ul} + \frac{\partial}{\partial x}(n \cdot \overline{ul^2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In these expressions the mean values $\overline{l}$, $\overline{ul}$, $\overline{ul^2}$ refer to those particles which at the time t are situated in the element Ω (between x and $x + dx$).

By means of these results eq. (18) can be brought into the form:

$$q = n \cdot u_s + u_s \frac{\partial}{\partial x}(n\, \overline{l}) - \frac{\partial}{\partial x}(n \cdot \overline{ul}) - \frac{\partial^2}{\partial x^2}(n \cdot \overline{ul^2}).$$

The last term, depending upon $\overline{ul^2}$, better can be neglected, as in making use of (15) we have already neglected quantities of the order l^2. Hence there remains:

$$q = n \cdot u_s + u_s \frac{\partial}{\partial x}(n \cdot \overline{l}) - \frac{\partial}{\partial x}(n \cdot \overline{ul}). \quad . \quad . \quad . \quad . \quad (21)$$

This equation can be used to obtain an estimate of the error which may be present in eq. (18*a*). It can also be applied to eliminate the part due to the systematic motion present in $\overline{ul}$. Indeed, writing:

$$u = u_s + u' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

we have:

$$\overline{ul} = u_s \cdot \overline{l} + \overline{u'l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and:

$$u_s \frac{\partial}{\partial x}(n\, \overline{l}) - \frac{\partial}{\partial x}(n \cdot \overline{ul}) = -n\, \overline{l}\, \frac{\partial u_s}{\partial x} - \frac{\partial}{\partial x}(n \cdot \overline{u'l}).$$

Hence (21) will become:

$$q = n\left(u_s - \overline{l}\, \frac{\partial u_s}{\partial x}\right) - \frac{\partial}{\partial x}(n \cdot \overline{u'l}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

9. It may be useful to recapitulate the meaning of the quantities occurring in the last equation.

In the first place n is the number of particles per unit volume, to be found at the instant of time t in the element Ω (between x and $x + dx$). These particles at that instant possess velocities u; in the preceding interval from $t_0 = t - T$ until t they had moved over distances l; both u and l will vary from particle to particle.

The "systematic velocity" u_s has been defined by (14*a*); it is the mean value of the velocities u at the instant t of the particles which at t_0 were

located in Ω, and thus it is not the mean value of the velocities of the particles which are to be found in Ω at t. The quantity u' has been defined as the difference $u - u_s$.

Finally $\bar{l}$ and $\overline{u'l}$ are mean values for the particles which are to be found in Ω at the instant t. The quantity $\bar{l}$ will differ from zero when a systematic motion is present. The fact that in eq. (24) there appears the combination

$$u_s - \bar{l}\,(\partial u_s / \partial x)$$

means that we should determine the "systematic velocity" rather for the element of volume from which, on the average, the particles started; it thus bears witness again to the unsharpness which is inevitable in these considerations.

10. In the preceding lines we have arrived at a direct deduction of the diffusion equation (24), which in cases where systematic motion is absent reduces to:

$$q_{\text{diffusion}} = -\frac{\partial}{\partial x}(n \,.\, \overline{u'l}) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

In this deduction no assumption has been made concerning a "mean free path" of the moving particles; the only hypotheses applied are

(1) that there exists a frequency function f, as defined in eq. (9);

(2) that it is legitimate to make use of the development (15), which in case of need perhaps might be extended.

For the elaboration of a theory of the diffusion of particles in a liquid or in a gas in turbulent motion, the next step now must be to deduce expressions for u and l, and thence for the mean values considered above, starting from data concerning the state of turbulence of the moving medium, and from data concerning the action of this medium upon the particles [3]).

Appendix. — It may be that the application, first of the transformation (15) or (16), then of the one occurring in (19) and (20), which seems to be of opposite nature, is somewhat confusing at first sight, and the reader may ask why these transformations do not cancel each others effect. The following formulae perhaps can render this point more clear; at the same time they indicate how the development can be extended.

According to (13):

$$q = \int dl \int du\, u\, F(x, l, u).$$

Instead of writing this formula in the form:

$$q = n\bar{u},$$

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[3]) A few provisional considerations concerning the latter point had been brought forward by the author in a discussional remark to a paper by PRANDTL; see A. GILLES, L. HOPF, und TH. VON KÁRMÁN, Vorträge aus dem Gebiete der Aerodynamik u.s.w. (Aachen 1929), pp. 8—10. These considerations, however, are not sufficient for further work and a quantitative investigation is necessary, which it is hoped will be undertaken by Mr. TCHEN CHAN MOU in collaboration with the author.

which is of no use for further work, we write:

$$F(x, l, u) = f(x, l, u) - l\frac{\partial F}{\partial x} - \frac{1}{2}l^2\frac{\partial^2 F}{\partial x^2} - \frac{1}{6}l^3\frac{\partial^3 F}{\partial x^3}\ldots .$$

Hence the current q can be expressed in the form:

$$q = n_0 u_s - \frac{\partial}{\partial x}(n\,\overline{ul}) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\,\overline{ul^2}) - \frac{1}{6}\frac{\partial^3}{\partial x^3}(n\,\overline{ul^3})\ldots .$$

This formula has the advantage that the systematic velocity u_s is brought into evidence; it has the disadvantage that in the first term there appears the quantity n_0, defined by (11a), which refers to the instant t_0. However, as we have:

$$f(x, l, u) = F + l(\partial F/\partial x) + \text{ etc.},$$

it follows that:

$$n_0 = n + \frac{\partial}{\partial x}(n\bar{l}) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\overline{l^2}) + \frac{1}{6}\frac{\partial^3}{\partial x^3}(n\overline{l^3}) + \ldots .$$

The formula for q consequently can be brought into the form:

$$q = n\,u_s + u_s\frac{\partial}{\partial x}(n\bar{l}) - \frac{\partial}{\partial x}(n\,\overline{ul}) + \frac{1}{2}u_s\frac{\partial^2}{\partial x^2}(n\overline{l^2}) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(n\,\overline{ul^2}) + \text{etc.}$$

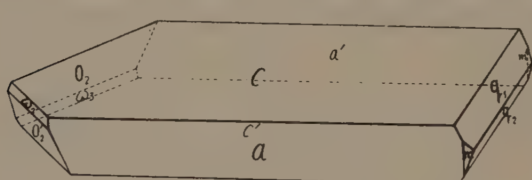
This is the extended form of eq. (21). The separation of u_s and u', in the way as was indicated by eqs. (22) and (23), finally will lead to the extended form of eq. (24).

Chemistry. — *On the Resolution of β-Phenyl-α-Propylene-diamine into its Optically-active Components.* (3th Part.) By F. M. JAEGER and W. FROENTJES.

(Communicated at the meeting of March 29, 1941.)

§ 1. In a previous communication [1]) the preparation and the properties were described of *β-phenyl-α-propylene-diamine*: $C_6H_5 \,.\, CH(NH_2) \,.\, CH(NH_2) \,.\, CH_3$; the base has a boilingpoint of 93° C. under a pressure of 2 or 3 m.M. In the present paper we publish the results of our tentatives to resolve this bivalent pseudo-base into its optically-active components. Although *four* optically-active components may theoretically be expected, we up till now have only succeeded to isolate *two* enantiomorphous isomerides in a pure state. Further investigations must enable us to decide, whether in the reduction-process applied also the two other possible enantiomorphous configurations are generated or not.

§ 2. The racemic compound was combined with so much of *d-tartaric acid* as to yield the mixture of the *mono-d-tartrates*. On repeatedly recrystallizing them from an aqueous solution, — as the less soluble product, — we obtained a fraction which finally showed a *constant* specific rotation $[\alpha]$ for sodiumlight of + 25°,5. The substance crystallizes from its aqueous solution at roomtemperature in beautiful, colourless and very lustrous, flat crystals, which have a typically hemimorphic aspect, as may be seen from the Fig. 1; they represent the *mono-d-tartrate* of the *levogyratory* base and contain *four* molecules of water of crystallisation.



Mono-d-Tartrate of l-C₆H₅-CH(NH₂).CH(NH₂)-CH₃.

Fig. 1. Crystalform of the *l*-Base-mono-*d*-Tartrate + 4 H_2O.

The crystals are *monoclinic-sphenoidal*; their axial ratio [2]), as deduced from the *X*-ray-spectrograms is: $a : b : c = 1{,}289 : 1 : 2{,}001$; with $\beta = 74°24'$.

Forms observed: $c = \{001\}$, predominant, very lustrous, occasionally finely striated parallel to the *b*-axes; $a = \{100\}$, smaller than *c*, but equally lustrous; $o_2 = \{\bar{1}\bar{1}1\}$, well developed and yielding ideal reflections;

[1]) F. M. JAEGER and J. A. VAN DIJK, these Proceedings, **44**, 26, 31 (1941).

[2]) The axial ratio directly calculated from the angular measurements is: $a : b : c = 1{,}261 : 1 : 2{,}003$; $\beta = 74° \, 24'$.

$q = \{011\}$, somewhat narrower than o_2, good reflecting; $\omega_1 = \{1\bar{1}0\}$ is ordinarily much narrower than o_2 and may even be completely absent, but it also can be as broad as o_2 and then yields very nice images; $m = \{110\}$, small but readily measurable; $s = \{0\bar{1}1\}$, small, yielding good reflections, but is often absent; $u = \{0\bar{1}2\}$, extremely small, in most cases absent. Some of the angular values appear to be rather variable ($\pm$ 15′ about their mean values). The habitus of the crystals is flat, elongated parallel to the b-axis and characteristically hemimorphic. A distinct cleavability was not observed.

Angular Values:		*Observed:*	*Calculated:*
$a : c$	$= (100) : (001) =$	*74° 24′	—
$c : q$	$= (001) : (011) =$	*62 36	—
$o_2 : o_2'$	$= (\bar{1}\bar{1}1) : (\bar{1}1\bar{1}) =$	*80 32	—
$a : o_2'$	$= (100) : (1\bar{1}\bar{1}) =$	58 31	58° 31′
$c : o_2$	$= (001) : (\bar{1}\bar{1}1) =$	76 55	76 55
$o_2' : \omega_1$	$= (1\bar{1}\bar{1}) : (1\bar{1}0) =$	22 58	22 56
$c : \omega_1$	$= (001) : (1\bar{1}0) =$	80 10	80 $9\frac{1}{2}$
$a : q$	$= (100) : (011) =$	97 14	97 $6\frac{1}{2}$
$a : m$	$= (100) : (110) =$	50 38	50 31
$q_1 : q_2$	$= (011) : (01\bar{1}) =$	54 48	54 48
$c : s$	$= (001) : (0\bar{1}\bar{1}) =$	62 36	62 36
$s : u$	$= (0\bar{1}1) : (0\bar{1}2) =$	18 36	18 38
$u : c$	$= (0\bar{1}2) : (001) =$	44 0	43 58

The axial plane is parallel to {010}.

From rotation-spectrograms round the a-b- and c-axis respectively, with Fe_α-radiation, the dimensions of the elementary cell were determined to be: $a_0 = 9{,}25$ Å; $b_0 = 7{,}30$ Å; $c_0 = 14{,}60$ Å (the latter controled by Dr. TERPSTRA). The elementary cell contains the mass of *two* tetrahydrated molecules. The density of the crystals at 17° C. was found to be: 1,304; calculated: 1,293. The spectrogram round the c-axis showed 6 spectra, that round the a-axis 5 spectra and round the b-axis equally so.

A SAUTER-spectrogram under 45° round the a-axis showed the following indices-triplets (TERPSTRA): (001), (003), (004), (005), (006); (011), (012), (013); (020), (021), (022), (023), (024), (025), (026); (031), (032); (040), (041); (044)
and one round the b-axis:
(001), (003), (004), (005), (006); (100), (101), (102); (104); (200), (201), (202), (206); (300), (301); (400), (401).

As $(h+k)$, $(k+l)$, $(h+l)$ and also $(h+k+l)$, appear to be even, as well as odd, *no* faces of the elementary cell are centred. According to

Dr. TERPSTRA, only the space-groups C_2^1 and C_2^2 can here be present; as (010) and (030) really occur, C_2^2 (resp. P_2^2) is here the only possible space-group. It is characterized by the presence of a single binary screw-axis; which is in accordance with the presence of 2 molecules in the cell, — the molecules being themselves asymmetrical.

The *d-tartrate* of the *l-base* yielded the following specific rotations in an aqueous solution of 2 % (see Table I).

TABLE I.
Some specific rotations of the *l*-Base-*d*-Tartrate.

Wavelength in Å:	Specific rotation [α]:
6480	+21°.0
6074	+23.2
5893	+25.5
5463	+30.3
5224	+33.1
5126	+34.8
4950	+37.0
4793	+41.8
4724	+42.5

§ 3. From the final mother-liquors the base was set free, the liquid destilled in vacuo and subsequently treated with *levo*gyratory tartaric acid and the mixture of the *l-tartrates* thus obtained subjected to fractional crystallisations in aqueous solutions, till a product was obtained which showed a *constant* specific rotation for sodium-light of $[\alpha]_D = -25°,5$. The crystals obtained had the same form of those just described; but they showed exactly the *enantiomorphous* development. ($c : q = (001 : 011) = 62° 37'$; $c : a = (001) : (100) = 74° 20'$; $q : q = (011) : (011) = 54° 47'$; $c : o = (001) : (111) = 76° 57'$. Axial plane = {010}.

Also their rotations were practically the same as those mentioned above, but with the opposite algebraic sign: the base set free from the *l-tartrate* is *dextro*gyratory *β-phenyl-α-propylene-diamine*. The anhydrous tartrates melt at 201° C.

The *d-* and *l-*bases thus obtained boiled at 110°—111° C. under a pressure of 8 mm. Their specific rotations at different wave-lengths are collected in the following table II.

The curves for the rotatory dispersion of the two pure bases are i.a. graphically represented in Fig. 2.

TABLE IIA.
Specific Rotations of the Optically-active β-Phenyl-α-Propylene-diamines.

Wave-lengths in Å:	Specific Rotations [α] of the Levogyratory Base:		Dextrogyratory Base:		Mean Values: (+ or —)
8050	—24°.8	—	—	—	(24°.8)
7607	—24.3	—	—	—	(24.3)
7280	—23.3	—	—	—	23.3
6980	—23.8	—	+24°.0	—	23.9
6730	—25.8	—26°.0	+25.4	+25°.8	25.8
6480	—28.1	—	+28.2	—	28.2
6262	—30.4	—30.4	+30.0	+30.3	30.3
6074	—32.6	—	+32.7	—	32.7
5893	—34.7	—34.9	+34.6	+35.0	34.8
5735	—37.1	—	+37.1	—	37.1
5592	—39.2	—39.3	+39.5	+39.7	39.5
5463	—41.5	—	+41.6	—	41.6
5340	—43.7	—43.7	+43.1	+44.2	43.7
5224	—45.8	—	+45.5	—	45.6
5126	—48.0	—48.1	+47.8	+48.6	48.1
3036	—50.2	—	+49.9	—	50.1
4950	—52.3	—52.9	+52.1	+53.1	52.6
4861	—54.4	—	+53.9	—	53.7
4793	—56.5	—	—	—	56.5
4724	—58.6	—	+58.2	—	58.4
4658	—60.5	—	+60.3	—	60.4
4596	—62.7	—	+62.6	—	62.7
4537	—64.9	—	+64.8	—	64.9
4483	—66.8	—	+66.8	—	66.8
4430	—68.9	—	+68.9	—	68.9
4380	—70.2	—	—	—	70.2
4335	—72.6	—	—	—	72.6
4290	—74.5	—	—	—	74.5
4248	—76.3	—	—	—	76.3

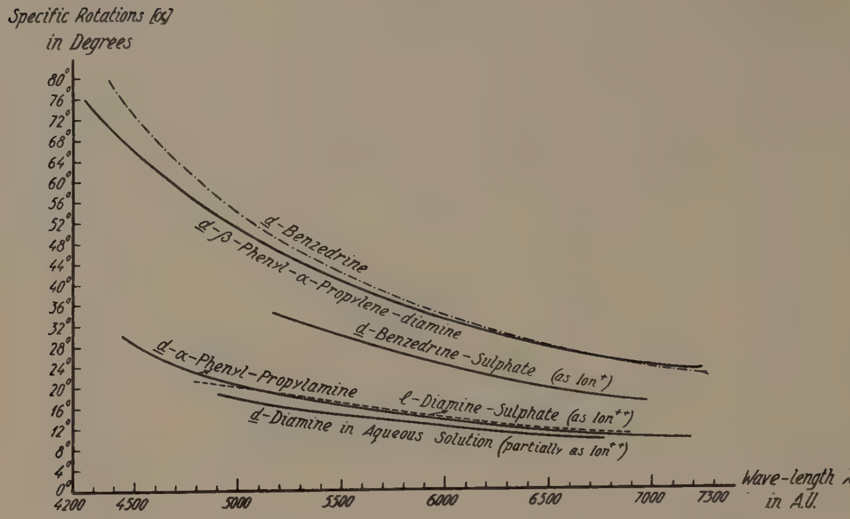


Fig. 2. Rotatory Dispersion of β-Phenyl-α-Propylene-diamine and its Ions; etc.

Simultaneously both pseudo-bases were also studied when dissolved in water; the measurements were made with solutions of 2,5 % of the *levo-* and of 2,1 % of the *dextro-*gyratory *diamine* (length of the tube: 20 cm). The specific rotations [α] thus observed were:

TABLE IIB. Rotatory Dispersion of the *d-* and *l-*diamines in aqueous Solutions.

Wave-lengths in Å:	Levogyratory Base: [α]:	Dextrogyratory Base: [α]:	Mean Values of [α]:
6730	$-9^\circ.4$	$+9^\circ.5$	$9^\circ.5$
6480	-10.2	$+10.2$	10.2
6262	-11.2	$+11.4$	11.3
6074	-12.0	$+11.9$	11.9
5893	-12.8	$+12.6$	12.7
5735	-13.2	$+13.3$	13.3
5592	-13.8	$+14.2$	14.0
5463	-14.4	$+14.7$	14.6
5340	-15.2	$+15.5$	15.4
5224	-16.0	$+16.2$	16.1
5126	-16.4	$+17.1$	16.8
5036	-17.2	$+17.9$	17.6
4950	-18.0	$+18.6$	18.3

A comparison of these values with the corresponding ones of the former table proves, that the rotations of the pseudo-bases in aqueous solution are for the same wavelengths only 0,4 to 0,33 of those of the pure bases themselves.

§ 4. In this connection it seemed of interest also to investigate the rotations of the aqueous solutions of the *salts* of the two antipodes with an optically-*in*active acid; in analogy to our similar measurements in the case of *benzedrine* and of *α-phenyl-propylamine* [1]), we for this purpose also here choose the neutral *sulphates*.

With respect to the fact, that from the *d-tartrate* we obtained the *levo*-pseudo-base and vice-versa, it could be expected, that also in the case of the *sulphates*, the *salts* would show rotations *opposite* to those of the pseudo-bases contained in them.

TABLE III. The Rotatory Dispersion of *d*- and *l*-Phenyl-*α*-propylene-diamine-Sulphates in Aqueous Solutions.

Wave-lengths in Å	Specific Rotations [α] of the Sulphate of the *l*-diamine (solution of 3.3 %):	Specific Rotations [α] of the Sulphate of the *d*-diamine (solution of 2.8 %):	Mean Values of [α] for the Ions: (+ or —)
6730	+11°.2	—10°.7	11°.0
6480	+12.0	—11.6	11.8
6262	+12.9	—12.5	12.7
6074	+13.6	—13.4	13.5
5893	+14.6	—14.5	14.5
5735	+15.2	—15.2	15.2
5592	+16.1	—16.4	16.3
5463	+17.0	—17.1	17.1
5340	+17.6	—18.0	17.8
5224	+18.3	—19.0	18.6
5126	+19.4	—20.0	19.7
5036	+20.3	—20.5	20.4
4950	+21.3	—21.4	21.3

The data of table III corroborate this conclusion: just as in the case of *α-phenyl-propylamine*, the *sulphates* derived of the optically-active *diamines* also manifest the *opposite* rotations in comparison with the pseudo-bases of which they are derivatives.

A highly curious fact is, that the rotation of the *dextro*-gyrate *sulphate*, — i.e. of the *l*-diamine-ion, — is *practically the same* as that of *d-α-phenylpropylamine* in the *un*dissociated state (Fig. 2).

1) F. M. JAEGER and J. A. VAN DIJK, loco cit., p. 40; F. M. JAEGER and W. FROENTJES, these Proceed., 44, 140 (1941).

In Table IV we also have collected the data obtained in the study of the rotations of the *sulphate* of *dextrogyrate stilbene-diamine*:

$$C_6H_5 . CH(NH_2) . CH(NH_2) . C_6H_5,$$

— a base previously [1]) studied in this laboratory. The salt was dissolved in water (0,57 %) and, therefore, the rotations observed equally correspond to those of the corresponding *ion*.

Although the curve between 5126 and 6480 Å. runs sensibly *parallel* to that of *d-β-phenyl-α-propylene-diamine* in the pure state, — the rotatory dispersion being, therefore, quite similar, — the difference of the rotations [α] between them still amounts to about 5°; the rotations themselves are about *twice* those of the *l-diamine-sulphate*.

TABLE IV. Rotatory Dispersion of *d*-Stilbenediamine-Sulphate in an aqueous Solution (0.5712 %).

Wave-lengths in Å:	Specific Rotations:
6730	+22°.4
6480	+24.0
6262	+25.4
5893	+29.3
5592	+34.2
5340	+39.0
5126	+43.3

With respect to the *β-phenyl-α-propylene-diamine-sulphate* here investigated, it can, moreover, be stated that the *absolute* values of [α], — although of *opposite* algebraic sign, — evidently are only slightly different from those characteristic for the free pseudo-base when dissolved in water, in which it is present, at least partially, in a dissociated state. As to the question whether really more *quantitative* relations between these different rotations do exist, we hope to return to it afterwards, as soon as all necessary data for this discussion will completely be available. —

The final mother-liquors of the various fractional crystallisations of the *tartrates* turn into a more or less viscous mass; we now are occupied in investigating them more thoroughly with the purpose of eventually being able to isolate the other isomerides which might possibly still be present in them.

Groningen, Laboratory for Inorganic and Physical Chemistry of the University.

[1]) J. G. BOS, *Thesis*, Groningen, (1940), p. 23 and 24. The base, being only insignificantly soluble in water, was investigated in its ethereal solution; its meltingpoint is 87° C.

Plantkunde. — *Bloemen of bollen bij Allium Cepa L.* II. Door A. H. BLAAUW, ANNIE M. HARTSEMA en C. W. C. VAN BEEKOM. (Mededeeling N⁰. 66 van het Laboratorium voor Plantenphysiologisch Onderzoek, Wageningen.)

(Communicated at the meeting of March 29, 1941.)

De opbrengst.

Terwijl op 8 April pas geplant werd, kon omstreeks 15—20 Juli geoogst worden. Tegen dien tijd beginnen de bladbundels bij den „hals", d.w.z. direct boven den bol, te knakken of om te vallen. De bollen, die grootendeels boven den grond staan, zijn gemakkelijk uit te trekken. Wij hadden pas 29 Juli gelegenheid het geheel te oogsten.

Feitelijk zijn alleen de bollen van niet-bloeiers, zoowel van één- als meerspruitige planten, van beteekenis voor het beoordeelen van de opbrengst. Intusschen zijn er ook bij de bloeiers nog wel enkele bruikbare bollen.

Bij de bloeiers kunnen zich velerlei gevallen voordoen: om den voet van een bloemstengel kan een bol gevormd worden, die wel van 40 tot 100 g kan wegen, maar in vele gevallen is die bolvorming zeer gering of geheel afwezig. Ook daar waar een flinke bol ontstaat heeft deze feitelijk geen waarde. Treedt de verhoutende stengelvoet uit het midden van den bol naar buiten, dan is de bol uiterlijk wel mooi rond maar er zit een holte in; treedt de stengelvoet meer zijwaarts uit den bol, dan is de bol ook uiterlijk misvormd. Hebben we nu met twee spruiten te doen, dan kunnen beide bloeien en doen zich dezelfde bezwaren voor. Bloeit slechts één spruit, dan kan ook hier de stengel min of meer zijwaarts uittreden en wel aan den kant van de andere spruit, zoodat dan de hier ontstane bol misvormd wordt. Treedt de bloemstengel uit het midden van de eerste spruit naar buiten, dan wordt de bol van de tweede niet-bloeiende spruit normaal rond. Alleen in dit geval geeft een bloeiende plant toch nog een volwaardigen bol. Het aantal van deze bollen is gering en het gewicht bedroeg in de verschillende groepen slechts 25 tot 45 g (zie ter vergelijking het gemiddelde gewicht der normale uien in tabel 5).

Om een beeld te krijgen van de opbrengst willen wij deze uitzonderingsgevallen buiten beschouwing laten en ons houden aan de opbrengst van de niet-bloeiende planten, die ook in de praktijk slechts meetellen.

Hier hebben we onderscheid te maken tusschen den oogst van eenspruitige en van twee- en meerspruitige planten. Wij willen hier vooruit reeds vermelden, dat *de „gekloofde" planten volkomen ronde bollen gaven, die zooals ook door ervaren kweekers erkend werd, practisch geheel gelijk-*

waardig bleken met de bollen van eenspruitige planten. Er is geen afplatting aan te zien en alleen aan den onderkant kan men in het midden herkennen dat het een tweeling-, resp. drielingbol is geweest.

De opbrengst wordt hier op twee wijzen gegeven. Allereerst naar de doorsnee in mm, omdat de waarde in de praktijk mede hierdoor bepaald wordt. De kleinste uitjes tot 35 mm, picklers genaamd, hebben nog vrij veel waarde daar ze voor inmaak gebruikt worden; dan volgt een grootte

TABEL 4.

Opbrengst der niet-bloeiers volgens doorsnee der bollen.

Telkens de eenlingen op de eerste rij en de tweelingen op de tweede rij.

(Yield of the non-bloomers arranged according to the diameter of the bulbs; first row of the singles; 2d row of the twins.)

Doorsnee in mm	Picklers				Gewone						Grove				
	0—30	30—35	35—40	40—45	45—50	50—55	55—60	60—65	65—70	70—75	75—80	80—85	85—90	90—95	95 enz.
A 5°	—	—	1	—	1	1	2	3	6	3	2	1	—	—	—
	2	1	–	3	4	1	1	1	1	–	–	–	–	–	–
B 9°	—	—	—	—	1	2	3	2	2	—	1	—	—	—	—
	2	6	4	3	5	2	3	1	–	–	–	–	–	–	–
C 13°	—	—	—	—	2	—	1	1	5	3	—	—	—	—	—
	1	–	2	2	7	4	2	3	1	–	–	–	–	–	–
D 17°	—	—	—	—	2	1	4	4	10	5	8	5	1	—	—
	1	3	1	4	9	10	8	10	4	–	–	–	–	–	–
E 20°	—	—	—	—	1	1	5	7	11	12	18	14	9	1	—
	–	–	5	8	14	23	21	17	12	9	3	2	–	–	–
F 23°	—	—	—	—	—	1	2	10	17	12	17	6	3	5	2
	1	1	5	6	16	30	27	18	11	8	4	2	–	–	–
G 25½°	—	—	—	—	—	1	1	4	14	21	15	6	8	7	—
	1	1	7	12	11	22	23	20	19	10	3	5	2	–	–
H 28°	—	—	—	—	1	1	2	9	18	11	17	18	9	4	—
	–	–	1	5	18	30	28	30	17	8	4	1	1	–	–
K 5°—23°	—	—	—	—	1	1	4	8	10	10	18	9	6	1	2
	–	1	1	3	8	12	9	16	12	4	4	2	1	–	–
L 9°—23°	—	—	—	—	—	1	2	5	13	12	13	10	3	5	—
	–	–	2	6	9	12	7	19	10	7	6	4	–	–	–
M 23°—5°	—	—	—	—	—	—	4	2	7	11	5	3	—	—	1
	–	–	–	2	3	5	5	–	3	–	2	1	–	–	–
N 23°—9°	—	—	2	1	—	1	3	4	6	7	5	4	1	1	—
	–	1	3	4	2	1	2	–	3	–	1	1	–	–	–
O 28°—5°	—	—	—	—	2	2	3	7	14	11	10	4	1	1	—
	–	2	8	3	11	7	4	6	1	2	–	–	–	–	–
P 28°—9°	—	—	—	—	—	4	5	9	11	14	8	9	1	—	1
	–	3	4	4	5	5	5	10	3	1	1	–	–	–	–

van 32—43 mm, die weinig waarde heeft; vervolgens 40 à 45 mm tot 70 à 75 mm, die als gewone maten bekend zijn, en boven 70 à 75 mm, die de grove uien genoemd worden.

In tabel 4 is de oogst gerangschikt naar de doorsnee, opklimmend met telkens 5 mm. Daarbij is in elke kolom aangegeven het aantal geoogste uien van die maat en wel eerst van de planten, die één bol leveren, en daaronder in kleine cijfers van de planten, die 2 bollen gaven.

Bij dit overzicht is reeds te zien, dat de bollen, die van de eenspruitige („ongekloofde") planten geoogst worden, gemiddeld grooter zijn dan de bollen, die twee aan twee ontstonden. Dit was ook licht te verwachten: bij tweespruitige planten wordt eerst het reservevoedsel van het moederbolletje gedeeld, verder ontplooien zulke spruiten 1 à 2 bladen minder dan een eenspruitige plant, en ten slotte moeten beide bollen door hetzelfde bodemgebied verzorgd worden. Aan den anderen kant zien we reeds in tabel 4, dat ook deze tweeling-uien voor het overgroote deel een maat bereiken, die het meest gezocht is (het meerendeel meet 50—70 mm doorsnee).

Het effect van de verschillende behandelingen is beter in tabel 5 naar het gewicht af te lezen.

In deze tabel is het gemiddelde gewicht per bol opgegeven, eerst van de bollen van éénspruitige planten (eenlingen) vervolgens van tweespruitige planten enz. In de laatste kolom is het totale gewicht van de geoogste uien van niet-bloeiende planten vermeld.

Zoowel in het aantal als in het gemiddelde gewicht staan de groepen E tot H (na 20° tot 28°) vooraan. Bij de tweelingen is ook het gemiddelde gewicht in de groepen K tot N hoog te noemen; het aantal is hier echter door de vele bloeiers te gering. In de meeste groepen brengt een plant met twee uien iets meer op aan gewicht dan een plant met één ui, vooral in groep G en K tot N.

De totale opbrengst van elke groep geeft een zeer duidelijk beeld van de uitwerking der behandelingen. De hoogste opbrengst wordt verkregen in H, na een behandeling met 28° C., vervolgens in G (25½° C.), — daarna volgen F (23°) en E (20°). De nauwkeurigheid van deze opbrengstcijfers wordt wel goed geïllustreerd door de bijna gelijke opbrengst van de naast-verwante groepen, zooals K en L met ruim 18 kg, O en P met 10 en 11½ kg, M en N met 6 à 7 kg, na de lage temperaturen 5° tot 13° (A—C) 2 à 3 kg. En evenals bij het percentage bloeiers staat weer D (17°) met 8½ kg tusschen de hooge en lage temperaturen in; dit was uit het aantal niet-bloeiers reeds te verwachten, maar men ziet hier dat ook het gemiddelde gewicht per bol tusschen de groepen A—C en E tot H in staat.

Voor de toepassing is dus uit het geheel der proeven duidelijk te concludeeren, *dat de plantuitjes van September tot half Maart in 25½° tot 28° C. bewaard, de beste uien met de hoogste opbrengst opleveren, waarbij de oogst reeds 15 à 20 Juli kan plaats hebben.*

Voor een bewaring in twee verschillende temperaturen (speciaal eerst koel, later warm) bestaat geenerlei reden.

TABEL 5.

Opbrengst der niet-bloeiers.
Gemiddeld gewicht per bol bij een-, twee- en drielingen.
Tusschen haakjes het aantal bollen.
(Yield of the non-bloomers; average weight of a bulb with singles, twins and triplets; bracketed the number of bulbs.)

	Eenlingen	Tweelingen	Drielingen	Totale gewicht
A 5°	(20) 112.3 g	(14) 44.6 g	—	2.870 kg
B 9°	(13) 112.8	(26) 45.4	—	2.668
C 13°	(12) 105.6	(22) 47.3	—	2.308
D 17°	(40) 130.1	(50) 67.5	(3) 34 g	8.680
E 20°	(79) 162.7	(114) 84.6	(6) 41.1	22.757
F 23°	(75) 162.8	(129) 82.6	(2) 44.3	22.957
G 25½°	(77) **164.0**	(136) **90.0**	(9) **65.7**	**25.455**
H 28°	(90) **162.8**	(143) **86.5**	(12) **94**	**28.154**
K 5°—23°	(70) 159.2	(73) 94.1	(9) 58.4	18.545
L 9°—23°	(64) 159.3	(82) 93.6	(6) 53.3	18.194
M 23°— 5°	(33) 141.6	(21) 84.9	(6) 45	6.726
N 23°— 9°	(35) 133.9	(20) 80.8	—	6.299
O 28°— 5°	(55) 136.2	(44) 58.6	(3) 29.2	10.159
P 28°— 9°	(62) 139.4	(41) 70.6	(3) 82	11.783

163 — 164 g
= ± 230 — 243 mm omtrek
= 73 — 74 mm doorsnee

83 — 94 g
= ± 180 — 192 mm omtrek
= 57 — 61 mm doorsnee.

Een volgend jaar worden deze proeven aangevuld door o.a. ter vergelijking in temperaturen beneden 5° C. te prepareeren, waarvoor eind Augustus 1939 in het laboratorium geen gelegenheid meer bestond. De Amerikaansche onderzoekers bevelen die lage temperaturen (—1° en 0° C.) aan om bloemvorming tegen te gaan. Het valt echter moeilijk aan te nemen dat — al wordt het bloei-percentage in hun proeven wel sterk gereduceerd — de opbrengst die van 25½° tot 28° C. zou kunnen evenaren. Wij wijzen er op, dat de opbrengst van de niet-bloeiers na 28° C. met 233 ronde bollen het 20-voud bedroeg van het gebruikte plant-materiaal van 200 uitjes (1,436 kg). Onze optimale opbrengst na 28° C. zou overeen komen met ± 45.000 kg per ha, terwijl in Hongarije een gemiddelde goede oogst op 25 tot 30.000 kg wordt gesteld. De hoogste opbrengst in één geval door THOMPSON en SMITH vermeld, bedroeg na bewaring bij 0° C. 23 kg van 200 geplante uitjes, maat 21—28½ mm. Van onze groep van 200 uitjes,

maat 19—26 mm, bleven 172 over om te planten en deze leverden 28 kg op, na bewaring bij 28° C. Men krijgt den indruk dat de Amerikaansche onderzoekers zich te spoedig van het beproeven van hooge temperaturen hebben afgewend. Volgens hun ervaring zouden de bolletjes daarin te veel verschrompelen en uitdrogen. Het is ook mogelijk, dat verschillende rassen zich hierin verschillend gedragen.

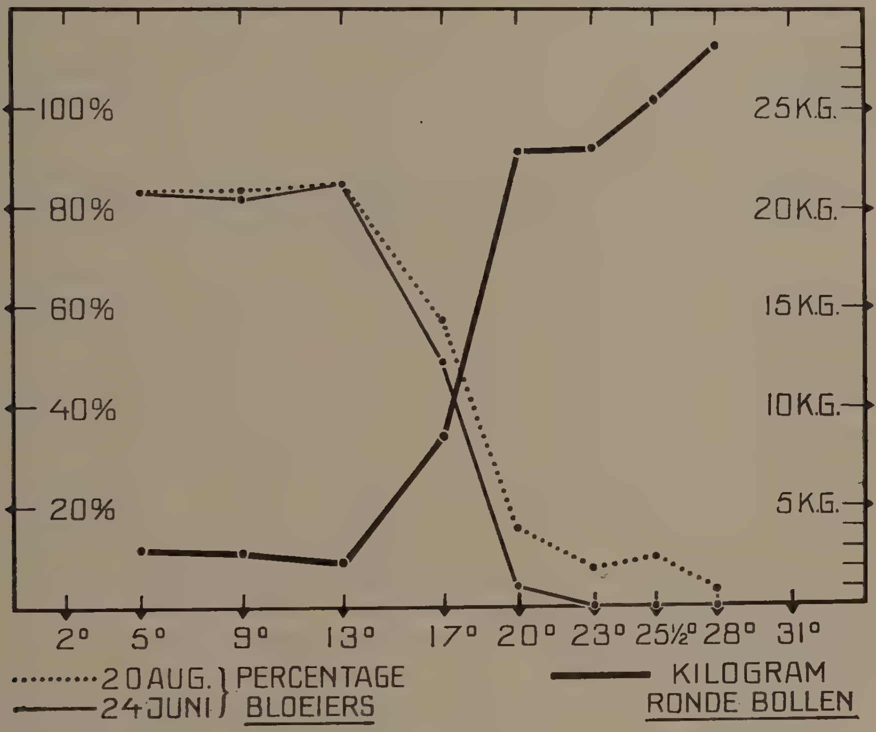


Fig. 1.

Slot.

Wij willen tot slot de tabellen 2 en 5, die ons een zoo scherpe uitkomst opleveren, nog een oogenblik in verband met elkaar beschouwen. Tabel 2 geeft ons met het percentage bloeiers een beeld van de generatieve functie na de verschillende behandelingen, terwijl tabel 5 in de opbrengst der planten, — het resultaat dus van bladontplooiing en assimilatie, — het beeld weerspiegelt van de vegetatieve functie. Beide beelden zijn als het ware elkaars complement. Figuur 1 doet dit voor 5° tot 28° ten overvloede nog eens duidelijk uitkomen, waarbij de scherpe omkeering boven 13° en beneden 20°, bij omstreeks 17° zeer in het oog springt.

In dit verband kunnen we wijzen op de Bol-irissen, waar ook 17 à 20° C. een keerpunt beteekent. Bloemvorming is bij deze Irissen alleen mogelijk *in* temperaturen tot ± 17° C. Bij 17° kan nog slechts een deel der bollen tot bloemvorming overgaan; bij een continue behandeling in 20° en hooger vindt geen bloemaanleg plaats. Nu hebben wij in dit artikel alleen gewezen

op het effect der temperaturen bij uien, zooals we dit als *nawerking* te velde constateerden. Daarbij viel op te merken dat in groepen, die bij 20° en hooger waren behandeld op den duur een klein percentage bloemstengels te voorschijn kwam. Zijn die bloemen echter in 20° enz. gevormd of pas na het planten bij de temperatuur van den bodem?

De invloed van de temperatuur op de bloemvorming zelf zal in een later artikel behandeld worden. Daarvoor werden ook reeds in dit jaar tijdens het bewaren op verschillende data bollen gefixeerd. Wij willen daaruit in verband met ons onderwerp den toestand vermelden op 19 Maart toen alle bollen uit de temperaturen naar 9° gingen en vervolgens geplant werden. Op dien datum is:

van 18 bollen uit 13° bij 15 de vorming van de bloeiwijze begonnen;

van 16 bollen uit 9° zijn 14 bloemvormend, iets minder ver dan in 13°;

van 18 bollen uit 5° zijn 14 bloemvormend, minder ver dan in 9°;

van 17 bollen uit 17° vertoonen 6 bollen een zwak begin van bloemvorming, terwijl in 20° tot 28° C. geen bloemvorming begonnen is. De bloemen die na 20° en 28° C. later nog te voorschijn traden, zijn dus in de bodemtemperatuur aangelegd.

In zoover vertoont de Ui groote gelijkenis met het gedrag der Irissen in die temperaturen. Of nu werkelijk *in* 20° tot 28° C. bij zeer lang verblijf bij de Ui geen bloem kàn gevormd worden zooals bij de Iris, is hiermee nog niet volkomen bewezen en wordt nog nader onderzocht. Daar het gedrag der Iris-bollen zoo sterk afwijkt van andere gewassen, speciaal van de ons bekende bloembollen, is de gelijkenis van *Allium Cepa* in dit opzicht van belang. *Het lage optimum van de bloemvorming bij 9° tot 13° C. komt bij Ui en Iris reeds volkomen overeen.* Deze vergelijking zal nader onderzocht worden.

De groepen die eerst hooge temperatuur ontvingen en in de 2e helft 5° of 9°, vertoonen op 19 Maart ten deele een begin van bloemvorming, zwak bij M (23°—5°) en P (28°—9°) en iets verder bij N (23°—9°); dit waren ook de groepen die (na 5° tot 13°) het rijkst bloeiden. K en L (eerst 5° en 9° daarna 23°), die op het veld veel minder bloeiden, vertoonen op 19 Maart geen bloemaanleg.

In figuur 1 is het percentage bloeiers zoowel op 24 Juni als op 20 Augustus afgezet. Wat er na 24 Juni bijkwam is in den bodem aangelegd en het is zeer onwaarschijnlijk, dat hierin nawerking van de tot 19 Maart gegeven temperaturen ligt. Daarom geeft de lijn van 24 Juni zeker een juister beeld van de werking en de eventueele nawerking der gegeven temperaturen.

Wageningen, December 1940.

FLOWERS OR BULBS WITH ALLIUM CEPA.

Summary.

For a biennial culture the onion is sown very densely in the first year, the small onions are gathered in the summer and then stored until the following spring. They are

then planted again and in the second half of July give the crop of full-grown onions. Now the danger is that they will flower and so not produce good bulbs. This depends on the treatment of the small bulbs. The object of the present investigation was to find out the influence of this treatment on their flowering or non-flowering.

In the first year onion sets were chosen for these experiments of the breed Zittauer Riesen, measuring 19—26 mm in diameter (equalling 6—8 cm in circumference) and with weights, varying from 6 to 9 g, a batch of 20 weighing 143,6 g. The usual size of such sets is 16—19 mm; one size bigger was chosen on purpose, because then the chance of flowers being formed is greater and the question was how to prevent this flower-formation by some treatment or other. The stock was received on August 29, 1939, and the experiments were started on September 4.

In the first place the bulbs were stored at temperatures, ranging from 5° to 28° C. (experiments A—H) until the time of planting. From practice, especially abroad it was known, that as a rule such small sets, in order to prevent flower-formation, are first kept cool till into December and then warm in a living-room. Therefore in the laboratory a number of groups were also treated with two temperatures, namely first cool (5° or 9° C.), and from December 18 warm (23° C.). As a check this order was also inverted, namely first warm (23° or 28°), followed on December 18 by 5° or 9° (experiments K to P, see the tables).

Each experiment was started with 200 selected bulbs (= 1436 g). These remained for 28 weeks, until March 1940, in those different temperatures. During this period in all groups some bulbs became diseased (soft or withered), and also when planted a few more were lost by shrivelling after insufficient rooting. In this way in 9 months' time from 10 to 25 percent of the bulbs were lost in the various groups; there was no clear evidence, however, of a relation to the treatment applied (see Table 1, of which the 4th column shows the number of dropped out bulbs in the course of a year, and the last column the loss of weight during keeping, in %).

On March 19, 1940, the temperature-treatment was stopped and all bulbs were then kept at 9° C., until they were planted in heavy clay under Maasdijk, which after the severe winter and subsequent rains could not be done before April 8. The distance between the rows was 20 cm, between the bulbs 10 cm. Four weeks later, on May 6, all groups had struck root well. No difference at all can be seen between the various treatments. A considerable number of plants show two shoots (are "split"), rarely 3 or 4 shoots (see below).

The flowering. After 6½ weeks (May 24) the first flower-stalks are visible, numerous in A (5°) and B (9°), still few in C (13°). Later the state of things develops which is rendered in Table 2 (percentage of bloomers). The higher the temperature of storing the fewer flower-stalks develop later. As late as July 9 the groups, which were stored at 23°, 25½° and 28°, do not show a single flower-stalk. After this date some bloomers still appear in the groups with no or few flowers, so that these groups also come out with a small percentage of bloomers. These flower-stalks, which are still very young when the bulbs are gathered, as a rule shrivel up entirely and do not harm the formed bulb any more.

Temperatures below 5° C. will be tried next year. American investigators prefer —1° and 0° C. for this treatment, while they disapprove of temperatures above 20° on account of drawbacks which for this breed we could confirm in no respect.

From the above it appears that in order *to prevent flower-stalks the onion sets must be stored at 23° to 28° C. (4 to ½ % bloomers on July 20).*

A treatment more or less in accordance with practical experience, "first cool, later warm" (K and L), does not give a large percentage of bloomers, but has no advantage whatever over "always warm".

The inverted treatment, tried as a check, does indeed yield a high percentage of bloomers.

The splitting. It could be expected that with these big-sized sets many "splitting"

plants would be found, i.e. plants with two or more shoots instead of one. What influence will the temperature-treatment have in this respect?

All in all there were on the field 1171 specimens with one and 1100 with 2 or more shoots. Table 3 gives the percentages of plants with one and with more shoots. Evidently the temperature cannot have had much influence.

As to the development of the onion, it was found on examination that in September, when the treatment was started, the buds of these shoots have already formed. None the less the treatment may of course have influenced the sprouting or non-sprouting.

The opinion, sometimes met with, that a high temperature would promote splitting, certainly does not hold good for the breed here used. It may be pointed out that among the plants with one shoot 443 were bloomers and 728 non-bloomers, among those with more shoots, 614 bloomers and 486 non-bloomers. So splitting plants clearly are more likely to flower.

Among agriculturists the opinion, or experience, is general that the bulbs of plants with more shoots ("splitting" plants) have a much lower selling-value. This point will be dealt with later.

The yield. Although planting took place as late as April 8, the bulbs could be gathered about July 15 or 20. Owing to circumstances it had to be delayed until Juli 29. The crop was laid out in the open and dried at Wageningen, and was measured and weighed on August 20. Also bloomers may sometimes produce more or less suitable bulbs, we shall restrict ourselves, however, to the yield of the non-blooming plants. Table 4 gives a survey of the diameters of the bulbs of non-splitting plants and under these, in small type, of the twins (two shoots, producing two bulbs).

It is seen from this table that more than half the bulbs of non-splitting plants, especially after warm treatment, have become big-sized (above 70 mm). The twins have for the greater part attained 50—70 mm, which is a very suitable size. Very few bulbs of 35—45 mm have been formed, which have little value.

Table 5 gives the average weight of individual, twin and triple bulbs (three shoots and therefore 3 bulbs with each plant), and in the last column the total yield of each group. This table shows very clearly indeed the favourable effect of the high temperatures, especially that of 28° C., which yielded most singles, twins and triplets with the highest total weight of over 28 kg, this being the crop of 200 small bulbs, (original weight 1,4 kg) of which 172 could be planted. In the columns of the singles and the total weights attention is drawn to the good agreement between similar treatments, e.g. 5°, 9° and 13° as contrasted with 20° to 28°; —17° being in every respect a transitional temperature, suited neither for the formation of seed nor of bulbs.

These big-sized sets yielded a crop with a large number of twins of very suitable dimensions. These split plants produced perfectly round bulbs, which experienced growers acknowledged to be practically quite equivalent to those of one-shooted plants.

In all respects the treatment at 25½° to 28° C. proves to give the best results for onion sets of this breed.

These phenomena with *Allium Cepa* are of importance for the relation between temperature and flower-formation. A nearer investigation on this subject is in course of progress. It has already appeared that on March 19, when the treatment was stopped, flower-formation had proceeded furthest with the bulbs in 13°, next in 9°, more feebly in 5° and hardly at all in 17°. As with the bulb-iris the optimum for flower-formation lies low, at 13° to 9° C. The flower-stalks which appear afterwards, after 20°—28° C., were not originated in that temperature, but in the soil after the planting.

The graph once more emphasizes the opposite effect of the lower temperatures, which here favour the generative processes, and of the higher temperatures, which inhibit the formation of flowers, while the vegetative activity is furthered and the yield of bulbs increased.

Anatomy. — *The oculomotor nucleus of the electric stargazers: Astroscopus guttatus and Astroscopus y-graecum.* By C. U. ARIËNS KAPPERS.

(Communicated at the meeting of March 29, 1941.)

Through the kindness of Prof. U. DAHLGREN of Princeton University N. J. we received, some years ago, the brains of two very interesting fishes: *Astroscopus guttatus* and *Astroscopus y-graecum*, occurring on the S. E. coast of America and in the Gulf of Mexico [1]).

As first observed by Dr. GILBERT for *Astroscopus guttatus* and by Dr. HENSHALL for *Astroscopus y-graecum* these fishes produce an electric shock when touched. According to Miss WHITE chemical stimulation has the same effect. Dr. GILBERT found their electric organs to be located behind the eyes [2]). As described by Prof. DAHLGREN in Science 1906, in the Carnegie Institute publications of 1914 and in an article in the Anatomische Anzeiger of 1906 by DAHLGREN and SILVESTER a large part of the eye muscles of these fishes has changed into an electric organ innervated by the oculomotor nerve.

Figure 1 is a copy of SILVESTER's drawing, showing on the right the extension of this organ and on the left the external muscles of the small eye (the obliquus inferior is not visible in this aspect).

From this drawing it appears that the muscles are very much reduced. The rectus inferior has even changed to a thin thread which only increases near its insertion on the eye ball.

All muscles, except the obliqui that have their insertions before the eye ball, are elongated to leave place for the electric organ.

According to these authors and to Miss WHITE, who examined its ontogenetic development the electric organ derives its tissue from the myotomes of all the eye muscles except the rectus inferior and obliquus inferior.

Although the obliquus superior, innervated by the trochlear nerve, and the rectus externus, innervated by the abducens, also participate in the structure of this organ, the organ, according to these authors, is innervated only by the oculomotor nerve by means of a strong branch, a small side-branch of which innervates what is left of the rectus superior muscle.

As appears from our copy of DAHLGREN's and SILVESTER's picture and

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1) We take this opportunity to express our thanks to Prof. DAHLGREN for this rare material.

2) Quoted from DAHLGREN and SILVESTER (so also JORDAN. A Guide to the study of Fishes, New York 1905 and JORDAN and EVERMANN. The fishes of North and Middle America. Bull. U.S. National Museum No. 47).

from our fig. 3 the oculomotor nerve is very big. Its transverse section is at least four times larger than that of the optic nerve.

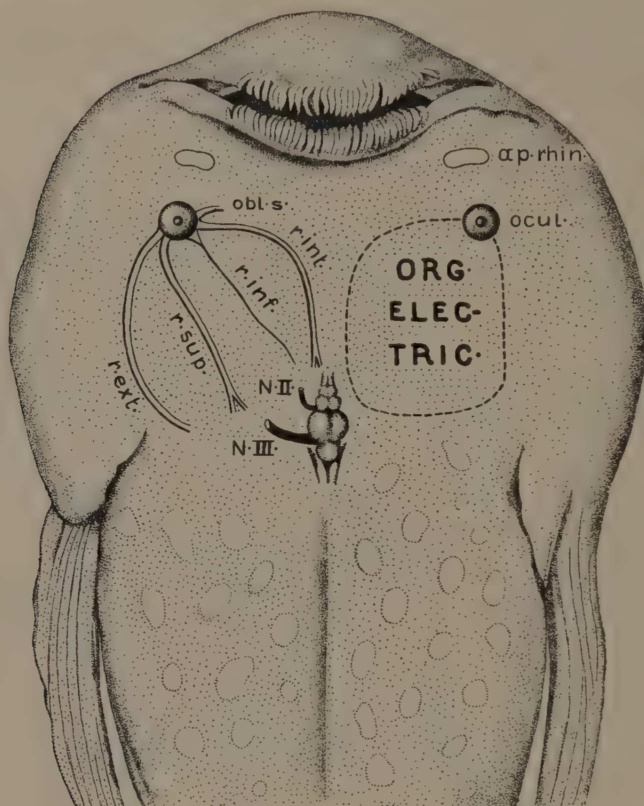

Fig. 1. Dorsal aspect of Astroscopus y-graecum, redrawn from DAHLGREN and SILVESTER. The m.m. obliqui of which only the m. obliquus superior is visible, are located frontally to the electric organ, the extension of which is indicated on the right side. The m.m. recti, originate behind the electric organ. The rectus superior runs through the electric organ and the rectus inferior is very much reduced. Compare the relative sizes of the oculomotor and optic nerves.

Although the electric nerve leaves the brain with the branch innervating the rectus superior, it apparently contains fibres from the whole oculomotor nucleus, as already observed by Miss WHITE.

Since the oculomotor nucleus of these fishes has not been described by these authors, we take this opportunity to give a brief description of its relations in Astroscopus guttatus and Astroscopus y-graecum. Unfortunately the state of preservation of the material, though quite sufficient for stating the general relations, did not allow us to study cytological details.

Our figure 2 shows the extension of the oculomotor nucleus of Astroscopus guttatus, projected on the sagittal plane of the brainstem, compared with an ordinary fish, Cottus scorpius, with normal eye muscles. From these diagrams, made in the way we introduced for measuring the locali-

zation and extension of motor nuclei, it appears that the oculomotor nucleus is enormously enlarged in Astroscopus guttatus. In Astroscopus y-graecum its metric relations are about the same.

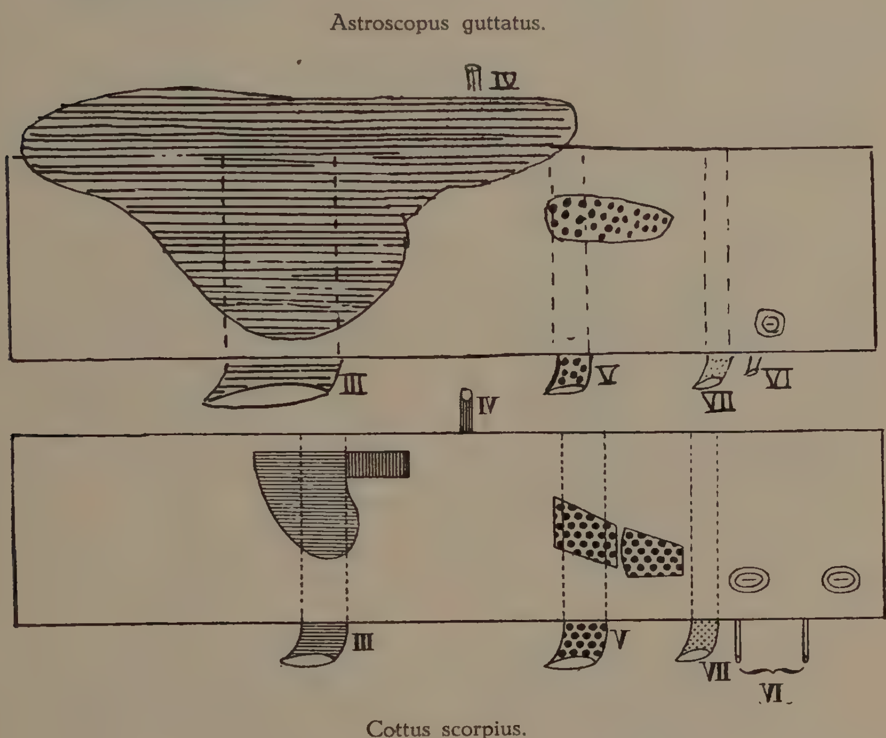

Fig. 2. Localization and sagittal extension of the eye muscle nuclei and roots and of the motor trigeminus nucleus and root in Astroscopus guttatus, compared with those of a fish, Cottus scorpius, with ordinary eye muscles. The oculomotor nucleus and root are horizontally, the trochlear nucleus and root vertically striped. The abducens nucleus is indicated by concentric circles, the motor trigeminus nucleus and root are dotted.

The oculomotor nucleus of Teleosts may be divided into a dorso-lateral and a ventral part (HUET, KAPPERS, BLACK, VAN DER HORST, ADDENS). The ventral part of the oculomotor nucleus which, according to HADIDIAN's and DUNN's experiments on the Goldfish (Carassius auratus), innervates the rectus superior (and obliquus inferior) in our Astroscopi is less hypertrophied than the dorso-lateral part.

From fig. 3 it appears that in *Astroscopus guttatus* the frontal pole of the dorso-lateral nucleus protrudes forward into the optic ventricle, its middle part lies between the valvula cerebelli, (val. cer.) and the nucleus lateralis valvulae (N.L.V.), and its hindpole lies above the sulcus limitans of the aquaeduct, occupying the same place as HERRICK's secondary gustatory nucleus does in other Teleosts, extending backward as far as the beginning of the motor trigeminus nucleus (Nu. V). The hindpoles

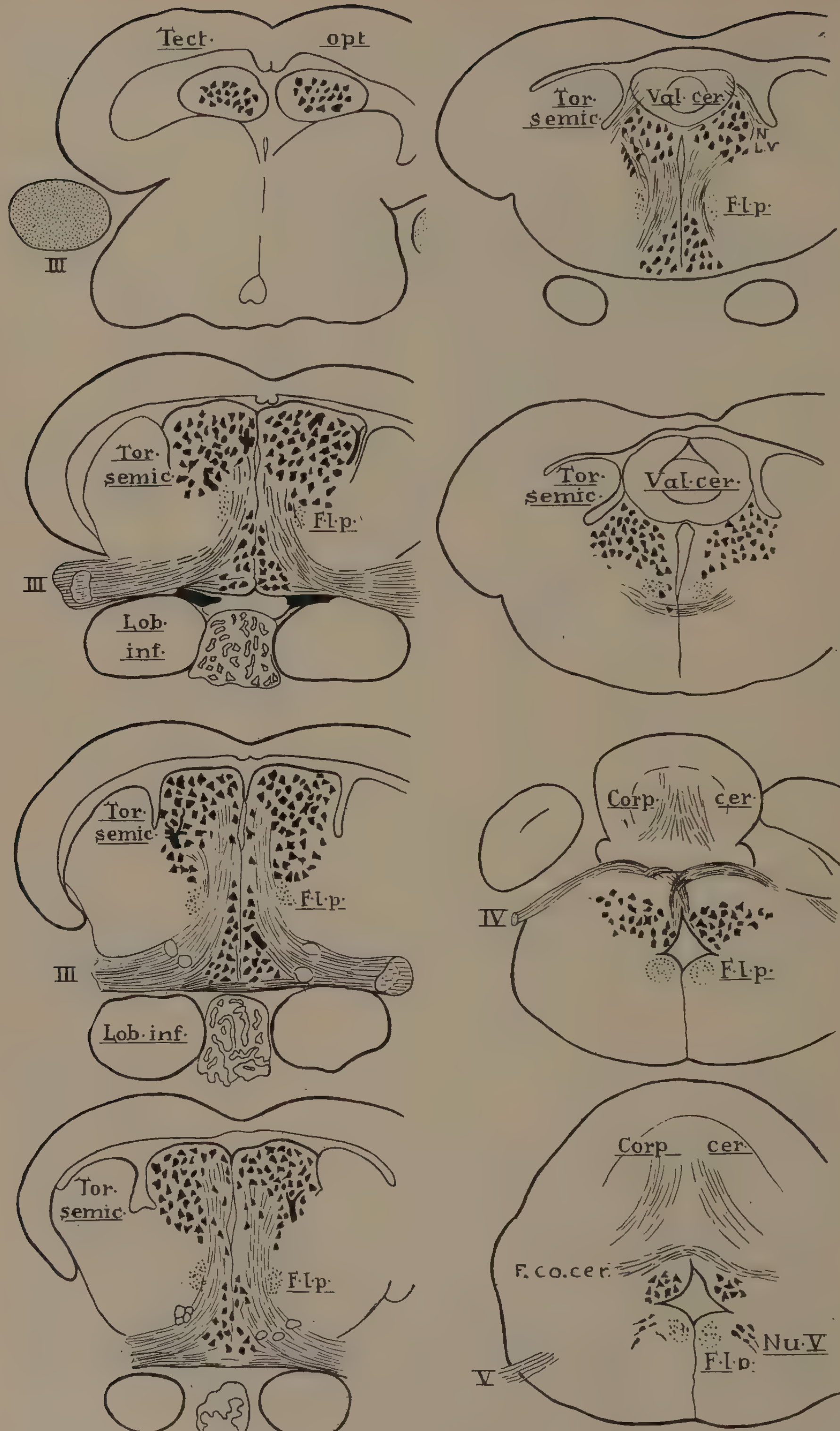

Fig. 3. Series of transverse sections of the oculomotor nucleus and trochlear region of Astroscopus guttatus.

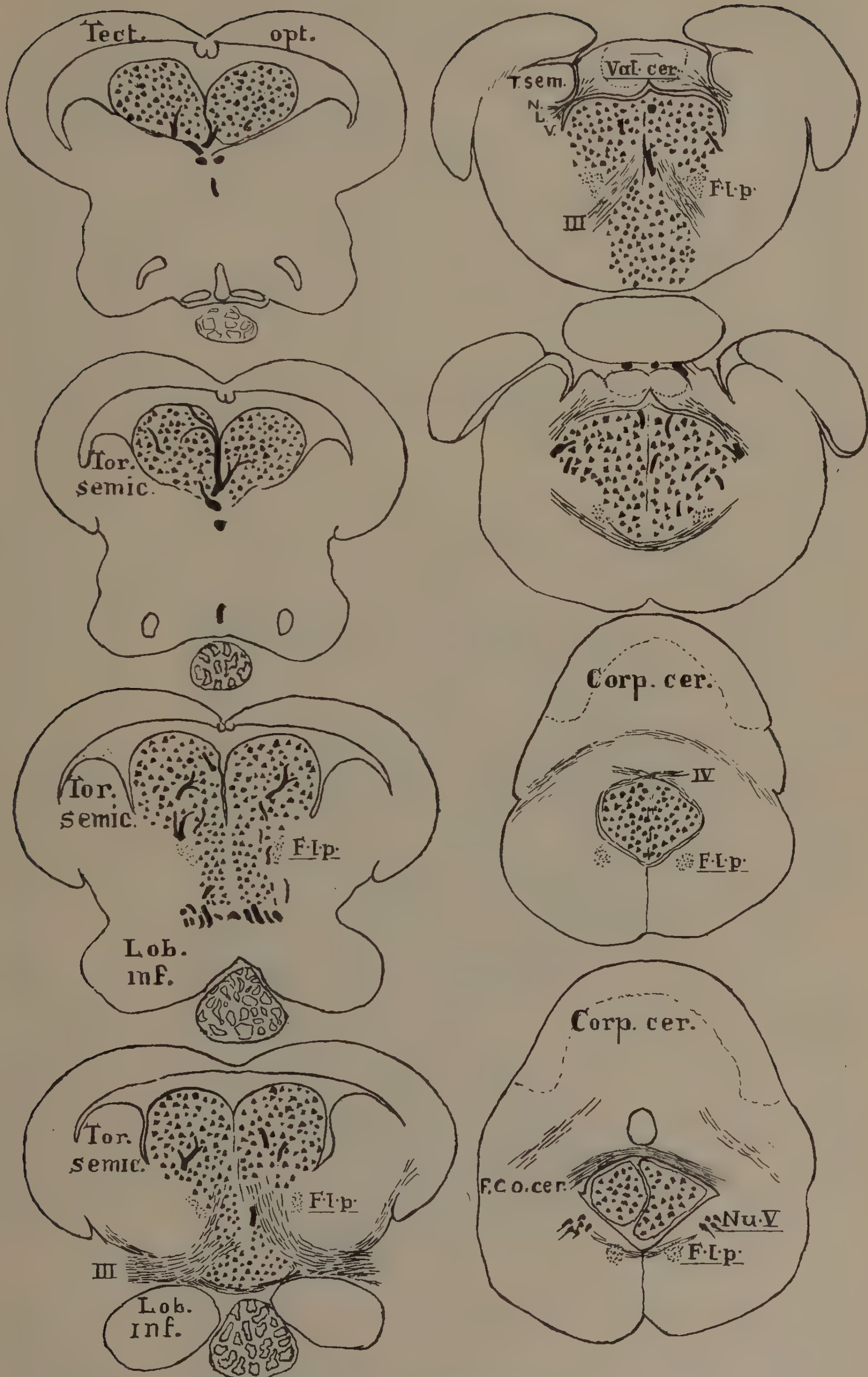

Fig. 4. Series of transverse sections of the oculomotor nucleus and trochlear region of Astroscopus *Y*-graecum.

of the nuclei of both sides are separated on this level by the dorsal extension of the aquaeduct.

In *Astroscopus y-graecum* (fig. 4) the hindpoles of the nuclei coalesce in the midline (as the dorso-lateral nuclei do), both halves being separated only by a thin septum. Together they practically fill up the whole aquaeduct, which at this spot is considerably extended by this nuclear mass.

This different morphology of the hindpoles in Astroscopus guttatus and Astroscopus y-graecum apparently influences the course of the trochlear fibres. In Astroscopus guttatus they run medially to the hindpole of the nucleus close to the ventricle. In Astroscopus y-graecum, they run laterally to the hindpole of the nucleus.

Although the trochlear roots are well preserved and have about the same position and size of the trochlear root in other Teleosts, the state of fixation in the centre of our material did not allow us to trace with certainty the origin of its fibres.

All we can say is that we have not been able to trace these fibres to a separate trochlear nucleus on the place where it usually occurs in Teleosts, i.e. on top of the fasc. longitudinalis posterior. Although this may be due to the state of our material, the relations in Astroscopus y-graecum suggest that the trochlear cells are located at the hindpole of the electric nucleus and that their celltype differs little from the type of the electric cells, although the cells are somewhat smaller.

The trochlear root, however, is not hypertrophied. In Astroscopus y-graecum it is even very small.

The abducens is not hypertrophied either, perhaps it is even reduced in size. The state of fixation not being any too good we have been able to find only one of its rootlets and by this we located one group of ventral cells belonging to it. There may be, however, a second rootlet and group of cells, as in most Teleosts.

The large size of the oculomotor nucleus is not due only to an increase in number of its cells but chiefly to their increase in size. The cells themselves closely resemble the cells of the electric nerves of Torpedo marmorata [1]) described by BORCHERT, KAPPERS, STUART and KAMP, and Narke Japonica [1]), described by SUZUKI. They are huge multipolar cells. Whether there are plasmodesms between these cells, as described by SUZUKI in Narke Japonica, we do not know. Another point of resemblance with the electric nucleus of the latter two fishes is its abundant vascularisation. Nearly each cell is located in a framework of capillaries [2]).

There are two questions which have to be considered in connection with our subject.

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1) The electric organ of these fishes is, however, derived from the gill musculature and innervated by the VII, IX and X nerves.

2) This is especially evident in our series of Astroscopus y-graecum, where the bloodvessels are still filled with blood. In our drawings only the larger bloodvessels are indicated.

The first question concerns those cells of the oculomotor nucleus that do not innervate the electric organ but the oculomotor muscles themselves, i.e. all the root cells of the rectus inferior and obliquus inferior (muscles which do not participate in building up the electric organ) and those root cells of the rectus superior and rectus internus, that send their neurites into these muscles.

The rectus inferior muscle being strongly reduced (see fig. 1), the search for its root cells (which in the goldfish are chiefly located in the dorso-lateral nucleus) is like looking for a needle in a bottle of hay. The root cells of the more developed obliquus inferior muscle according to HADIDIAN's and DUNN's experiments occur chiefly in the ventral nucleus. It may be therefore, that the somewhat smaller and darker cells of the ventral nucleus, indicated in our microphotograph of *Astroscopus y-graecum* (fig. 5), innervate this muscle. Since, however, most cells of the rectus superior and about

Fig. 5. Transverse section about the middle of the oculomotor nucleus of Astroscopus y-graecum. Nu. III d.l. = dorso-lateral, nu. III V = ventral part of the nucleus; bl. v. = bloodvessels.

half of the cells of rectus internus according to these authors are also located in the ventral nucleus, the somewhat smaller cells located here may as well serve the rectus superior and internus.

Curiously enough the purely motor cells of the III nucleus are also larger than the III cells of an ordinary Teleost and the same may hold good for the cells of the trochlear nucleus, if our assumption that the latter are located at the hindpole of the electric nucleus is right.

Considering the general reduction of the eye muscles it is not astonishing that the number of these purely motor cells is small.

The second question is inspired by the fact that, whereas the myotomes of the trochlear and abducens muscles participate in the formation of the electric organ, the trochlear and abducens nerve do not participate in its innervation. Apparently the elements derived from the obliquus superior and rectus externus myotomes joining the electric organ are innervated by oculomotor branches.

This heterogenous innervation might be explained by an overlapping of the termination of the three eye muscle nerves.

Anastomoses of oculomotor fibres with the trochlear and abducens ending in the obliquus superior and rectus externus muscles have been described by POIRIER and CHARPY for man. MURAT and RUPASSOW experimentally confirmed their occurrence in cats. This overlapping, which only implies a few fibres, may be compared with the plurisegmental innervation of extremity muscles experimentally proved by AGDUHR and confirmed by LAWRENTJEW.

Another explanation, advocated by ADDENS, is based upon the phenomenon of *secondary heterotopic innervation*, as also observed in the tail musculature, where myotomes may be innervated by a root different from the root normally innervating that myotome.

Since peripheral anastomoses between the eye muscle nerves of Teleosts have not been observed, the latter explanation seems the more likely one.

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Medicine. — *Détournements des différentes formes de nystagmus dans le système nerveux central.* II. Par A. DE KLEYN.

(Communicated at the meeting of March 29, 1941).

C. *Détournements de l'arc réflexe observés dans certaines affections du système nerveux central sans cause déterminée.*

Contrairement aux observations précédentes, nous avons pu étudier nombre de cas de détournements sans avoir pu en préciser clairement l'étiologie chez des malades atteints le plus souvent d'affections nerveuses localisées dans la fosse cérébelleuse.

I. *Le détournement du nystagmus rotatoire s'observe le plus souvent.* Dix cas de ce genre ont pu être observés [1]).

A. Dans 9 cas nous avons vu un nystagmus horizontal paradoxal chez des malades atteints d'affections bien différentes: 5 cas souffrant du syndrome de *Ménière,* 2 cas de tumeur de l'angle ponto-cérébelleux, une séquelle de méningite pneumococcique et un cas d'atrophie du cervelet.

B. Dans un cas nous avons pu observer après l'épreuve rotatoire dextrogyre un nystagmus vertical battant vers le bas et après l'épreuve lévogyre un nystagmus horizontal au lieu d'un nystagmus rotatoire normal. Le diagnostic de probabilité de ce cas serait un processus pathologique se développant dans l'étage moyen-gauche du crâne (dr. STENVERS). Voici en résumé deux observations cliniques de tumeur de l'angle ponto-cérébelleux (cas de la série *A*).

a. premier cas, Homme, âgé de 35 ans.

Une tumeur de l'angle ponto-cérébelleux droit diagnostiquée par le dr. STENVERS était lors de l'opération confirmée par le neurochirurgien. Le malade succomba d'une pneumonie quelques jours après l'opération.

L'examen auditivo-vestibulaire, fait un an avant l'intervention chirurgicale, montrait encore un fonctionnement partiel du nerf acoustique et aucune anomalie des pyramides pétreuses n'était à voir sur la radiographie.

Examen auditif:

Oreille droite: surdité type oreille moyenne.

Diapason de 16 à 40 V.D. et diapason du ut (64 V.D.) n'étaient pas entendus.

Audition nettement raccourcie du Fa^6.

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[1]) NEUMANN, Jahrbuch f. Psychiatrie, **36**, 550 (1914) a vu dans un cas de tumeur du velum medullare superius un nystagmus horizontal post-rotatoire au lieu d'un nystagmus rotatoire. On observe souvent dans des affections du système nerveux central l'absence isolée du nystagmus rotatoire postgyratoire. Ces cas, ainsi que l'absence du nystagmus postrotatoire observé chez les nouveaux-nés seront rapportés plus tard.

Epreuve de Rinne négative.
Epreuve de Schwabach raccourcie.
Voix chuchotée: ad concham.

Oreille gauche: normale.
Examen vestibulaire:
Troubles spontanés:
En regard direct: léger nystagmus spontané horizontal vers la gauche.
En regard latéral gauche: idem.
En regard vers le bas: idem.
En regard latéral droit: nystagmus spontané horizontal vers la droite.
En regard vers le haut: nystagmus vertical vers le haut.
Nystagmus de position: présence dans les deux décubitus:
En position latérale droite de la tête: nystagmus spontané horizontal vers la droite.
En position latérale gauche de la tête: nystagmus spontané horizontal vers la gauche.
Signe de l'indication spontanée: bras droit déviation spontanée vers le bas, le bras se mouvant dans le plan horizontal.

Epreuve calorique:
Oreille droite: eau froide: tant pour des quantités d'eau faible (5 cc.) et massives (75 cc.): pas de nystagmus, pas de signe de l'indication réactionnelle. Déviation des yeux vers la droite.
Eau chaude: légère réaction nystagmique vers la droite.
Oreille gauche: eau chaude et froide: réactions nystagmiques normales. Signe de l'indication réactionnelle: normal.

Epreuve rotatoire (disque électrique)
1. En position assise:
10 tours dextrogyres (temps 33″): nystagmus post-rotat. horizontal vers la gauche, 27 secousses en 14″. La durée fut difficile à déterminer avec exactitude par la présence d'un léger nystagmus spontané vers la gauche.
10 tours lévogyres: (temps 33″): nystagmus post-rotat. horizontal vers la droite, 29 secousses en 13″.
2. *En décubitus dorsal*:
10 tours dextrogyres (temps 33″): *nystagmus post-rotat. horizontal*: 9 secousses en 7″.
10 tours lévogyres (temps 33″): *nystagmus post-rotat. horizontal*: durée du nystagmus 12″.
3. En décubitus latéral droit:
10 tours dextrogyres (temps 33″): nystagmus post-rotat. vertical dirigé vers le bas, 19 secousses en 12″.
10 tours lévogyres (temps 33″): nystagmus post-rotat. vertical vers le haut, 20 secousses en 9″.

b. Le deuxième cas se rapporte à une femme âgée de 52 ans, atteinte d'une tumeur de l'angle ponto-cérébelleux.

L'examen auditif révéla une surdité totale de l'oreille gauche et une fonction parfaite de l'oreille droite. L'examen radiographique des os temporaux (fait le 6 mai 1936) montre une grande dilatation du conduit auditif interne gauche avec atteinte du bloc osseux environnant.

Examen vestibulaire:
Troubles spontanés:
Nystagmus spontané battant vers le haut seulement le regard porté vers le haut. Pas de nystagmus de position.

Epreuve calorique:
Le labyrinthe droit réagissait bien et normalement à l'épreuve chaude et froide. Signe de

l'indication réactionnelle présente (déviation des bras dans la direction typique aux deux épreuves).
Le labyrinthe gauche ne réagissait pas à l'épreuve chaude et froide non obstant l'emploi de quantités croissantes d'eau chaude et d'eau froide.

Epreuve rotatoire (disque électrique).
1. Malade en position assise:
10 tours dextrogyres (30″): 21 secousses; nystagmus post-rotatoire vers la gauche.
10 tours lévogyres (30″): 18 secousses en 11″: nystagmus post-rotat. horizontal vers la droite.
2. *Malade en décubitus dorsal:*
10 tours dextrogyres (30″) 21 secousses en 10″: *nystagmus post-rotat. horizontal* (non rotatoire).
10 tours lévogyres (30″): 12 secousses en 17″: *nystagmus post-rotat. horizontal* (non rotatoire).
3. Malade en décubitus latéral droit:
10 tours dextrogyres (30″): 25 secousses en 12″, nystagmus post-rotat. vertical vers le bas.
10 tours lévogyres (30″): 16 secousses en 11″, nystagmus post-rotat. vertical vers le haut.

Dans ce cas nous voyons une absence totale de nystagmus spontané en regard direct, constaté dans le cas précédant.

c. Le troisième cas est le rapport d'une fillette âgée de 13 ans chez qui lors d'une méningite pneumococcique nous avions exécuté une labyrinthectomie droite en date du 21 janvier 1939. Pas de nystagmus spontané à voir dans les différentes positions du regard. A l'épreuve rotatoire (18 mars 1940) nous pouvions déclencher un nystagmus post-rotatoire horizontal faible mais typique battant vers la droite et la gauche ainsi qu'un nystagmus post-rotatoire vertical dirigé vers le haut et le bas. Pendant ces formes de nystagmus nous pouvions voir cependant une déviation prononcée des globes dans le sens de la phase lente. *Aucune tentative à déclencher un nystagmus post-rotat. gyratoire ne nous réussissait, mais une simple déviation horizontale apparut après une rotation lévogyre et dextrogyre, la malade étant couchée sur le dos.*

d. Le cas suivant montre un exemple d'un détournement simultané d'un nystagmus rotatoire en un nystagmus horizontal et d'un nystagmus vertical battant vers le haut en un nystagmus horizontal.

Un homme, âgé de 46 ans, était sujet à des crises violentes de vertiges, accompagnées de vomissements multiples.

Examen interne et neurologique: normal (27 décembre 1939).

Examen sérologique: réactions de BORDET–WASSERMANN et de SACHS–GEORGI négatives.

Examen auditif: normal (voix chuchotée et examen aux diapasons).

Examen vestibulaire:

Troubles spontanés: pas de nystagmus spontané, pas de nystagmus de position. Le nystagmus optocinétique est normal dans toutes ses formes: cortical et sous-cortical dans le plan horizontal et vertical.

Epreuve calorique: normale (eau chaude et froide).
Réactions d'adaption statique de RADEMAKER et GARCIN: vives.

Epreuve rotatoire (disque électrique):
1. Malade assis:
10 tours dextrogyres (48″), 43 secousses en 43″: nystagmus post-rotat. horizontal vers la gauche.
10 tours lévogyres (50″): 36 secousses en 39″: nystagmus post-rotat. horizontal vers la droite.

2. *Malade en décubitus dorsal*:
10 tours dextrogyres (48"): 20 secousses en 13": *nystagmus post-rotat. horizontal vers la droite.*
10 tours lévogyres (50"): 10 secousses en 9": *nystagmus post-rotat. horizontal vers la gauche.*
3. *Malade en décubitus latéral droit*:
10 tours dextrogyres (48"): 16 secousses en 10": nystagmus post-rotat. vertical vers le bas.
10 tours lévogyres (50"): 10 secousses: *nystagmus post-rotat. horizontal vers la gauche* (absence de nystagmus vertical vers le haut).

Lors des examens, recommencés peu après dans les mêmes conditions techniques, le malade étant couché sur le dos mais tourné dans le sens lévogyre, nous avons pu observer cependant quatre secousses nystagmiques post-rotat. gyratoires vers la gauche, suivies immédiatement de 16 secousses horizontales vers la gauche. Après la rotation dans le sens dextrogyre, la réaction restait identique au premier examen. Dans notre série seulement un cas analogue (secousses nystagmiques post-rotatoires, suivies de secousses horizontales) nous est connu encore.

Le même phénomène se présentait dans notre observation suivante au moment où nous voulions déclencher un nystagmus vertical.

e. Cinquième cas. Un homme âgé de 35 ans, souffre de crises de vertiges pendant lesquelles tout semble tourner, accompagnées de sensations nauséeuses désagréables.

Examen interne et neurologique: normal.

Examen radiographique: (21 mars 1933): normal. Configuration des conduits auditifs internes: normale.

Audition: voix chuchotée: oreille droite, entendue à 75 cm; oreille gauche, entendue à 25 cm (surdité du type labyrinthique).

Examen vestibulaire:

Troubles spontanés: absence de toutes les formes.

Epreuve calorique (chaude et froide): les deux labyrinthes semblent normaux. Le labyrinthe gauche est un peu plus excitable que le droit. Signe de l'indication réactionnelle normale, à l'exception de l'indication vers l'intérieur du bras gauche.

Epreuve rotatoire (disque électrique).
1. Malade assis.
10 tours dextrogyres (30"): 52 secousses en 25", nystagmus post-rotat. horizontal vers la gauche.
10 tours lévogyres (30"): 30 secousses en 20": nystagmus post-rotat. horizontal vers la droite.
2. *Malade en décubitus dorsal.*
10 tours dextrogyres (30"): 50 secousses en 26", *nystagmus post-rotat. horizontal vers la gauche,* pas de nystagmus rotatoire vers la droite.
10 tours lévogyres (30"): 36 secousses, *nystagmus post-rotatoire vers la droite, suivi de 18 secousses: nystagmus horizontal vers la droite,* le tout en 33".
3. *Malade en décubitus latéral droit:*
10 tours dextrogyres (30"): 17 secousses en 14": *nystagmus post-rotat. vertical vers le bas, suivi de quelques secousses horizontales vers la gauche.*
10 tours lévogyres (30"): 30 secousses en 16": *nystagmus post-rotat. vertical vers le haut, suivi de 2 secousses horizontales vers la droite.*

Après ces seuls exposés cliniques, je ne tarde pas à signaler deux faits:

1. Dans la série des neuf cas, deux cas seulement présentent un léger nystagmus spontané dans le regard direct (un cas de tumeur de l'angle ponto-cérébelleux et un cas de Ménière). Les sept autres cas cependant ne présentent absolument aucun nystagmus spontané.

2. Un point important est le fait que dans les cas où l'on voulait déclencher un nystagmus post-rotat. gyratoire p.ex. vers la gauche (malade couché en décubitus dorsal), celui-ci n'était pas seulement remplacé par un nystagmus horizontal, mais ce dernier prenait une direction tantôt vers la gauche, tantôt vers la droite.

Des 9 cas, trois cas présentaient un nystagmus horizontal vers la gauche et quatre cas un nystagmus vers la droite. Dans deux cas le sens du nystagmus horizontal n'a pas été enregistré.

L'inverse se montre lors de la rotation lévogyre. Ce fait démontre que dans certaines affections du système nerveux central, les détournements ne se présentent pas seulement par des modifications dans les formes nystagmiques réactionnelles mais au surplus par des modifications dans le sens du nystagmus déjà modifié.

g. Homme, âgé de 30 ans.

Diagnostic probable du neurologue dr. STENVERS: Affection cérébrale de l'étage moyen-gauche.

Examen acoustique (21 janvier 1935): normal. Voix chuchotée entendue des deux côtés à la distance normale.

Examen vestibulaire:

Troubles spontanés:

Absence de nystagmus spontané.
Absence de nystagmus de position.
Absence du signe de l'indication spontanée.

Epreuve calorique (chaude et froide).

Les deux labyrinthes sont normaux et le signe de l'indication réactionnelle: typique.

Epreuve rotatoire:

1. Malade en position assise:

10 tours dextrogyres (31"): 38 secousses en 21". Nystagmus post-rotat. horizontal vers la gauche.

10 tours lévogyres (31"): 29 secousses en 14": nystagmus post-rotat. horizontal vers la droite.

2. *Malade en décubitus dorsal*:

10 tours dextrogyres (31"): 31 secousses en 18": *nystagmus post-rotat. vers le bas (au lieu d'un nystagmus rotatoire vers la droite).*

10 tours lévogyres (31"): 27 secousses en 16": *nystagmus post-rotat. horizontal vers la droite (au lieu d'un nystagmus rotatoire vers la gauche).*

3. Malade en décubitus latéral droit:

10 tours dextrogyres (31"): 40 secousses en 15", nystagmus post-rotat. vertical vers le bas.

10 tours lévogyres (31"): 16 secousses en 8", nystagmus post-rotat. vertical vers le haut.

Dans ce cas un nystagmus vertical, battant vers le bas, remplaçait le nystagmus rotatoire classique après rotation dextrogyre et un nystagmus horizontal détourne le nystagmus rotatoire après la rotation lévogyre.

II. *Dans les deux cas suivants le nystagmus horizontal est détourné en un nystagmus vertical après rotation des malades assises, la tête légèrement inclinée en avant* [2]).

a. Premier cas, Madame L., âgée de 72 ans, souffrant de fortes névralgies du trijumeaux et de crises de vertiges, est admise à la clinique. Le docteur STENVERS, neurologue, ayant assisté personnellement à deux de ces crises, avait pu observer un nystagmus rotatoire spontané battant vers la droite lors de la première et un nystagmus spontané horizontal lors de la deuxième crise, mais sans aucune perte de conscience. Il y avait pendant la marche une chute bien nette vers la droite (dysharmonie vastibulaire de BARRÉ).

Les examens internes et neurologiques étaient normaux.

Signalons qu'un an auparavant la malade avait souffert d'une lithiase rénale et avait subi une ablation de la glande mammaire (carcinome).

Examen acoustique (29 octobre 1934): normal.

Examen vestibulaire:

Troubles spontanés: pas de nystagmus.

Epreuve calorique: n'a pas eu lieu par suite d'un eczéma du conduit auditif externe.

Epreuve rotatoire:

1. *En position assise.*

10 tours dextrogyres: pas de nystagmus.

10 tours lévogyres: *nystagmus post-rotat. vertical battant vers le bas,* durée 13″.

2. En décubitus dorsal:

10 tours dextrogyres: nystagmus rotatoire vers la droite, durée 18″.

10 tours lévogyres: nystagmus rotatoire vers la gauche, durée 18″.

A la suite de l'état très nauséeux et vertigineux de la malade, le nystagmus vertical n'a pu être contrôlé.

Son départ nous a empêché de faire un examen plus complet. Par la suite je peux rapporter que deux ans plus tard la femme semblait être débarrassée de ses souffrances.

b. Deuxième cas. Chez un homme, âgé de 38 ans, souffrant d'une sclérose multiple (Dr. STENVERS) (15 mai 1933) nous avons pu recueillir les données cliniques suivantes:

Examen acoustique: normal.

Examen vestibulaire:

Troubles spontanés:

En regard direct: pas de nystagmus.

En regard latéral droit: nystagmus horizontal spontané vers la droite.

En regard vers le haut: nystagmus spontané vertical dirigé vers le haut.

En regard vers le bas: nystagmus spontané vertical dirigé vers le bas.

En regard latéral gauche: parésie du regard, les yeux ne dépassant pas la ligne médiane. Pendant la tentative des globes oculaires de porter le regard vers la gauche, il n'y a aucun nystagmus à voir.

Pas de nystagmus spontané, pas de signe de l'indication spontanée.

Epreuve calorique: l'épreuve calorique froide de l'oreille gauche, le malade en décubitus

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2) JONES (JONES, I. H., Equilibrium and vertigo, Lippincote, Philadelphia and London 1918) signale dans ce travail différents cas de „perverted nystagmus" entre autres: un cas d'un malade atteint d'une tumeur pontine présentait un nystagmus rotatoire post-gyratoire au lieu d'un nystagmus horizontal.

UNTERBERGER (Ztschr. v. Hals-, Nasen und Ohrenheilk, **36**, 207 (1934), a observé un cas clinique d'un tubercule localisé dans la fosse Rhomboïde qui présentait un nystagmus vertical post-rotatoire au lieu d'un nystagmus horizontal.

dorsal, déclenchait un nystagmus horizontal vers la droite, accompagné du signe de l'indication typique.

L'épreuve calorique froide de l'oreille droite, ne provoquait qu'une déviation des yeux dans le plan horizontal vers la droite et le signe de l'indication réactionnelle.

Epreuve rotatoire:

1. *Position assise:*

10 tours dextrogyres (32″) *nystagmus post-rotatoire vertical vers le bas,* pas de nystagmus horizontal, 14 secousses en 16″.

10 tours lévogyres (32″) déclenchent un *nystagmus post-rotatoire horizontal vers la droite, suivi* peu après par un *nystagmus diagonal.*

2. En décubitus dorsal.

10 tours dextrogyres (32″): nystagmus post-rotatoire vers la droite, 23 secousses en 13″.

10 tours lévogyres (32″): nystagmus post-rotatoire vers la gauche, 5 secousses en 8″.

3. En décubitus latéral gauche.

10 tours dextrogyres (32″): nystagmus post-rotatoire vertical vers le haut, 7 secousses en 8″.

10 tours lévogyres (32″): nystagmus post-rotat. vertical vers le bas, 50 secousses en 17″.

L'épreuve rotatoire recommencée et contrôlée quelques jours plus tard, nous donna les mêmes résultats.

L'examen de la forme subcorticale du nystagmus optocinétique nous révéla les mêmes anomalies. Dans le but de déclencher la forme subcorticale battant vers la gauche dans le plan horizontal, nous voyions un nystagmus vertical dirigé vers le bas. La forme corticale battant vers la droite dans le plan horizontal était normale.

Ce cas nous montre d'une façon bien nette combien les détournements dans le système nerveux central peuvent être isolés. Ce malade p. ex. réagissait normalement à l'épreuve calorique par un nystagmus horizontal et présentait à l'épreuve rotatoire et optocinétique horizontale un nystagmus vertical paradoxal.

Medicine. — *Tonic neck-reflexes on the eye-muscles in man.* By A. DE KLEYN and H. W. STENVERS.

(Communicated at the meeting of March 29, 1941.)

In 1907 BÁRÁNY [1]) in his publication on "Augenbewegungen durch Thoraxbewegungen ausgelöst" communicated that when fixing the head of rabbits and then turning the trunk in relation to the head round different axes, eye-movements developed. These new eye-positions remained as long as the position of the trunk in relation to the head was not changed. Later it became evident that these neck-reflexes on the eyes are of great physiological significance [2]), at least as far as animals like rabbits which have no spontaneous eye-movements are concerned.

The first communication of these reflexes in man was of BIKELES and RUTTIN [3]). They recorded the history of an 18 year-old male, who developed complete deafness after recovering from a cerebrospinal meningitis. It was impossible to elicit vestibular reflexes by caloric stimulation, turning etc. However, marked compensatory eye-movements developed after turning the head in relation to the trunk. The authors mentioned different reasons to explain their opinion that here reflectoric and not voluntary eye-movements existed. The cause of these eye-movements was thought to be either vestibular reflexes (which had to be elicited in another part of the labyrinth than the caloric and turning reactions — e.g. in the otolithic apparatus —) or reflexes which were caused by stimulation of the neck joints. To discriminate these possibilities they performed the following experiment. They fixed the head in relation to the trunk and then turned both head and trunk 90° to the right or to the left. No eye-movements developed behind the closed eyes, whereas these could very well be observed when the head was turned in relation to the trunk. So they draw the following conclusion: "Demnach muss man die bei activen und passiven Kopfdrehungen beim Patient festgestellten reflektorisch, kompensatorischen Augenbewegungen als nicht vestibulärer Ursprungs, sondern als durch sensibele Reize infolge Lageveränderungen in den entsprechenden Gelenken ausgelöst ansehen".

From the communication of BIKELES and RUTTIN it appears that they

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1) BÁRÁNY, R., Zentralbl. f. Physiol., **20**, 298 (1907).

2) KLEYN, A. DE, Arch. Néerl. de Physiol. de l'Homme et des animaux 2, 644 (1918) et **7**, 138 (1922); Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 509 (1920); Pflügers Archiv, **186**, 82 (1921).

3) BIKELES, G. und RUTTIN, E., Neurol. Zentralbl., **34**, 807 (1915).

did not examine exactly if in their case there existed compensatory eye-movements or compensatory eye-positions. Also in later literature this difference is not always taken into consideration, and owing to this much confusion arose.

Compensatory *eye-movements* develop after movements of the head and immediately or nearly immediately stop when the head ceases to move. They are reflexes of the semi-circular canals and are called compensatory because the eyes are drawn in a direction opposite to that of the head, thus trying to keep their position in space.

On the other hand compensatory *eye-positions* stop as soon as the head alters its position in space or in relation to the trunk. They remain unchanged as long as the positions of the head or the trunk are the same [1]).

In daily life the position of the head is incessantly changed: compensatory eye-*movements* and compensatory eye-*positions* coöperate in a harmonious way. During these changes in position in the first place movements are made by the head which, when not executed too slowly, are accompanied by compensatory eye-movements. Secondly both the position of the head in space and the position of the head in relation to the trunk are changed. The tonic neck and labyrinthine reflexes on the eyes caused in this way give rise to compensatory eye-positions.

The name "tonic" was given to these last reflexes because they stand out as long as the positions of the head and trunk remained unchanged; the word "labyrinthine reflexes" was used when the reflexes were caused by changing the position of the head in space (i.e. a change of the labyrinths in space); "neck reflexes" if the reflexes were due to a change of position of the head in relation to the trunk.

We have at our disposal the following auxilliary methods to examine the tonic labyrinthine and tonic neck-reflexes separately.

Labyrinthine reflexes can be examined separately by bringing the head in different positions in space, taking care that the position of the head in relation to the trunk remains unchanged. In men or animals with eliminated labyrinths, neck-reflexes develop when the position of the head is changed in relation to the trunk, or when, in men or animals with a functionating labyrinth, the development of labyrinthine reflexes is prevented by fixing the head, and the position of head and trunk in mutual relation is changed by moving the trunk.

Besides, no tonic labyrinthine reflexes appear in the various asymmetric positions of the head in relation to the trunk if the patients turn their head in sitting position, incline their head in dorsal position or move their head back and foreward in lateral position. This is due to the fact that in all

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[1]) Generally speaking compensatory eye-*movements* develop if the animal or individual makes movements with the head from out the *normal* position. The same holds good with regard to the compensatory eye-*positions* developing after turning the head from the *normal* to another position (DE KLEYN l.c.).

these cases the position of the labyrinths in relation to the horizontal plane remains unchanged.

It was BÁRÁNY [1]) who first made this investigation in man. In neonati and premature infants he changed the position of the trunk in relation to the fixed head and observed a change of the eye-positions in the direction of the palpebral-fissure. The eye-positions remained as long as the position of the trunk in relation to the head was unchanged.

These horizontal tonic neck-reflexes on the eyes could be observed by BÁRÁNY only during the first two days after birth; after this spontaneous eye-movements of the children made the observation impossible.

These spontaneous eye-movements account for the fact that in adults the reflexes could only be demonstrated in cases of serious diseases of the central nervous system, that is to say as far as tonic neck reflexes in horizontal and vertical direction are concerned. Rotatory eye-positions, caused by tonic labyrinthine and neck-reflexes, can also be observed in normal individuals.

This, however, is not surprising considering the fact that voluntary rotatory eye-movements cannot be executed and so cannot have a disturbing influence.

We shall yet give the different forms a more extensive notice.

A. Horizontal eye-positions caused by tonic neck reflexes.

As communicated above BÁRÁNY observed these reflexes in neonati and premature infants. In elder patients we were able to make the following observation.

B., 12 years.

This patient was examined by us in the clinic of Professor WINKLER. Some photographs of the changed eye-position after turning of the head have already been published elsewhere [2]).

Here follows a more extensive history together with the pathological-anatomical disturbances in the central nervous system.

Previous history (May 31st, 1920): In August 1919 the patient complained of bad sight, head-ache and vomiting. The gait became unsteady: on admission it was impossible for her to walk. Speech was slurring, but the patient did not use the wrong words.

Present history: June 6th, 1920: When in rest a strabismus convergens was present, the right eye deviating to medial. Permanent large vertical and horizontal movements of the eyes were present but no nystagmus. Looking to the left was not maximally, at the same time a quick, nearly vertical nystagmus with its quick component upward to the left appeared.

Looking to the right was also restricted, the left eye moves more in the nasal direction

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[1]) BÁRÁNY, R., Acta Oto-Laryng., 1, 97 (1918); VOSS (Folia Oto-Laryng., II, 24, 16 (1925)) and BERBERICH und WIECHERS (Ztschr. f. Kinderh., 38, 59 (1924)) described such reflexes in neonati.

[2]) STENVERS, H. W., Ztschr. f. d. ges. Neur. und Psych., 92, 484 (1924); Handb. d. Neurologie, Buke Foerster, Bnd V, 543 (1936); MAGNUS, R., Körperstellung, Berlin, Springer (1924).

than the right one in the direction of the temporal bone; now an irregular, sometimes vertical, sometimes horizontal nystagmus was present.

Looking up- and downward is coordinated and maximally; the vertical nystagmus was respectively directed upward and downward. The sensation for touch was disturbed on the right arm, trunk and leg.

The feeling of forms was disturbed at the right and at the left. There existed a right-sided paresis of the extremities together with a strong tendency for various movements with large components.

Both the left and right papillae nervi optici were atrophic. The visus was $< 1/60$, on the left side less than on the right. The pupils reacted to light.

When trying to walk the patient showed a strong tendency to walk and fall backward. If the patient was supported in the axilla during her attempt to walk, the head and the trunk were inclined to the right.

Juni 30th, 1920: *Ventriculography*: 200 cc. of air were insufflated: both ventricles appeared to be considerably enlarged. On the left side, at the level of the gangliae, a large shadow was seen.

March 1921: A strong analgesia of the right part of the trunk could be observed. Deep sensation was absent on the right side.

April 1921: A motoric aphasia had developed.

April 29th, 1921: The patient was examined with Prof. MAGNUS. On turning the head to the left and to the right, the right arm showed distinct typical tonic neck-reflexes, more pronounced on right than on left turning.

February 23th, 1922: Syringeing of the left ear with 55 cc. of water (28° C) caused a pronounced nystagmus to the right. Syringeing of the right ear with 55 cc. of water (28° C): immediately a horizontal nystagmus to the left appeared.

February 28th, 1922: Examination with Professor MAGNUS: *typical strong tonic neck reflexes, both on the arms and on the legs.*

January 20th, 1923: Head in dorsal position, symmetrically in relation to the trunk. Eyes were in the middle position and made permanent irregular nystagmoidal movements.

On turning the head to the right, the right eye moved maximally to the left, the left eye somewhat to the right.

The same appeared if the movements of the head were executed slowly to eliminate reflexes of the semi-circular canals.

If the patient was placed in the right lateral position, the head symmetrically in relation to the trunk (fig. 1) the position of the eyes was completely the same as described above (head in dorsal position, symmetrically in relation to the trunk). If however, the trunk was turned to the dorsal position but the head was left in the right lateral position, the right eye returned maximally to the left and the left eye somewhat to the right (fig. 2).

The reverse took place if, the patient in dorsal position, the head was turned to the left or, the patient completely in the left lateral position, the trunk was turned to the dorsal position (head remaining in left lateral position). Now the left eye moves maximally to the right and the right eye somewhat to the left. These eye-reflexes were of a tonic character and remained as long as the position of the head and the trunk was unchanged.

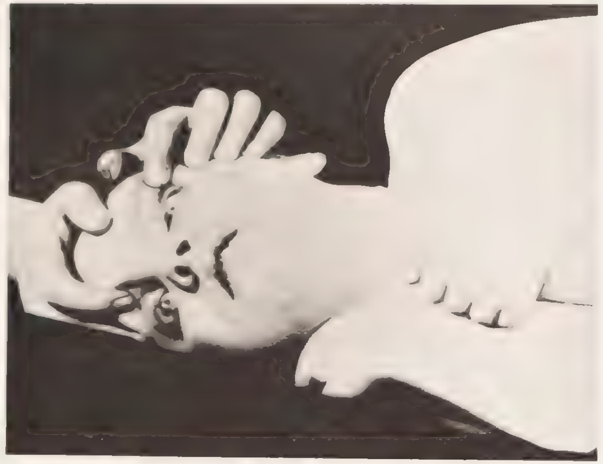


Fig. 1

Fig. 1a

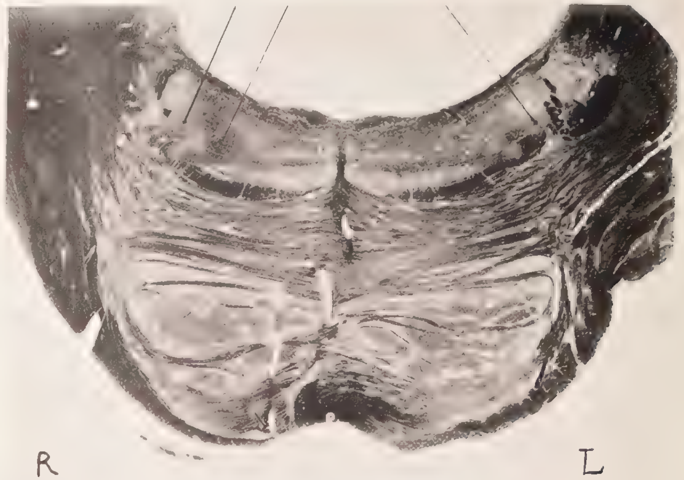
Fig. 2.

Tumor in left basal ganglia

Fig. 3

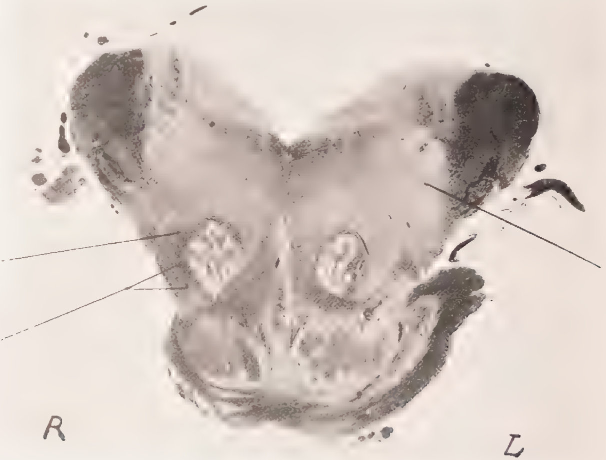
Tumor protruding in aquaeduct

Fig. 4



Section at the level of the motoric trigeminal nerveroot

Fig. 5



ction through frontal part of olive

Caloric examination: 75 cc. of water (17° C), right ear: a horizontal nystagmus to the left of the *right eye* developed. This eye overreaches in the slow phase the middle position to the right.

On the *left eye* a horizontal somewhat rotatory nystagmus to the left appeared, the eye came to the middle position during the quick phase, but did not surpass it.

75 cc. of water (17° C): *Left ear*: Strong, long-lasting horizontal nystagmus to the right of both eyes.

December 31st, 1923: *Right ear*: Syringeing with cold water elicited first a maximal deviation of the right and left eye to the right. After this a strong horizontal nystagmus of the *right* eye to the left appeared. *Left eye*: A horizontal rotatory nystagmus to the left appeared, somewhat directed downward: the eye reached scarcely over the middle position.

Left ear: Syringeing with cold water: a strong horizontal nystagmus to the right of both eyes appeared. During the slow component to the left, the left eye did not overreach the middle position, the right eye moved quite undisturbed to the left.

Eye movements:

Looking upward: The patient raised her eye-brows, at the left more than at the right. The eye-lids were also moved, however *not trace of any movement whatever of the eyes upward was to be seen.* In the eyes repeated convergential movements appeared. Often in the left eye a horizontal movement directed to the right was present, whereas in the right eye a rotatory movement with its upper pole directed to the right appeared, at the same time the right eye moved somewhat downward.

Looking downward: The left eye turned horizontally to the right. The right eye turned downward.

Looking to the right: The left eye turned to the right, the right eye moved to the middle position to the right, and showed nystagmus movements directed downward.

Looking to the left: Patient moved her head to the left direction, in the eyes however no looking to the left could be noticed.

July 17th, 1923. The patient. †

On autopsy the ventricles appeared to be extremely dilated. The left basal gangliae were enlarged and protruded in the dilated left ventricle. The cerebellum and the pons were macroscopically normal.

Frontal section of the basal ganglia (fig. 2) revealed a voluminous cystic tumour in the left basal ganglia. The right thalamus and the basal ganglia were displaced to the right. On serial examination it appeared that the tumour was continuous at the left side through the mesencephalon. The tumour protruded in the aquaeducts. The tumour was tightened by the brachia conjunctiva. The left brachium conjunctiv. was partly proliferated by the tumour. The tumour infiltrated and destroyed the frontal part of the substantia nigra; distally and medially the nearly intact border of the left nucleus was present.

In the same section at the right side, the nucleus ruber was damaged more distally. A remnant of the corpus subthalamicum can still be seen.

The fibres of the nucleus ruber were stronger on the left than on the right. In the oculomotor nucleus, displaced to lateral, intact cells are distinctly seen on the right. On the left the cells have nearly completely disappeared.

On the left side of the motoric trigeminal nerveroot, in the tegmentum, the number of transverse sectioned fibres is but small (fig. 4). On the right the pallido-tegmental and the pallido-olivary tracts (nomenclature WINKLER) are distinctly seen; on the left they have disappeared. On the right in the rubro-spinal tract there is atrophy. Section through the frontal part of the inferior olivary nucleus (fig. 5) reveals a marked degeneration of the left pallido-olivary and pallido-tegmental tracts.

The crossed, i.e. the right rubro-spinal tract is also nearly absent.

With regard to the extension of the tumour and the degenerations of the damaged tracts it is evident that on the left side the basal ganglia have been eliminated until and including the frontal part of the mesencephalon just above the nucleus ruber.

The degeneration of the rubro-spinal tracts proves that the left nucleus ruber has been damaged seriously.

B. *Vertical eye-positions caused by tonic neck reflexes.*

In 1924 SCHUSTER [1]) published an interesting observation of a patient who developed, when looking downward, a complete ocular palsy, whereas the eye-movements in all other directions were normal. Sometimes the patient was able to follow an article, provided this object was moved downward from the upper part of the visual field.

The following interesting symptoms were noted:

If the head of the patient was inclined with the chin on the chest, the eyes turned upward (patient in dorsal or in standing position). If, however, the head was turned in the opposite direction, the eyes went downward. When doing so it was necessary to distract the patient's attention. SCHUSTER called this symptom the "Puppenkopfphänomen".

This experiment could be made more easily if the movements were done quickly. SCHUSTER considered this to be due to the fact that the development of disturbing eye-movements was more difficult in quick movements. Therefore spectacles, covered with paper, were placed before the eyes, to avoid fixation of the eyes.

To settle the fact whether or not the above mentioned reflexes were of labyrinthine origin, the following experiment was made. The patient was fixed on a horizontal plank so that the position of the head in relation to the trunk could not be changed. Now the upper part of the plank was slowly turned downward, the spectacles being placed before the eyes.

Even on turning the plank more than 45°, the eyes remained immovable, as could be noticed from side-face. Turning of the plank in the opposite direction revealed the same phenomenon.

From these experiments SCHUSTER draws the conclusion that, in his case, the just-mentioned reflexes were neck and not labyrinthine reflexes.

SIMONS [2]) draws attention to the fact that it was not stated by SCHUSTER if the reflexes on the eyes were tonic, i.e. if the eyes remained deviated as long as the head was turned in relation to the trunk. He communicated also, that together with BERNHARDT [3]), he had examined in 1919 an unconscious patient with encephalitis lethargica, which patient developed, six days before exitus, when bending the head on the chest (which could be done without pain) a marked regular and rather quick deviation of the eyes. The eyes remained tonically in this new position as long as the position

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[1]) SCHUSTER, P., Dtsch. Ztschr. f. Nervenheilk., **70**, 97 (1921).

[2]) SIMONS, A., Ztschr. f. d. ges. Neurol. und Psych., **80**, 499 (1923).

[3]) BERNHARDT, G. und SIMONS, A., Neurol. Centralbl., **39**, 705 (1919); SIMONS, A., Neurol. Centralbl., **39**, 132 und 256 (1920).

of the head was unchanged. Besides the patient showed the same symptoms when being in dorsal, sitting or lateral position. BERNHARDT and SIMONS did not investigate if in their patient tonic neck or tonic labyrinthine reflexes were present, because, when making their examination, they had no knowledge of these two forms. However, it is evident that in their case tonic neck-reflexes were present, as the symptoms developed in sitting, dorsal and lateral position of the patient.

It is important to remember that SCHUSTER emphasises the fact *that the reflexes on the eyes developed especially when moving the head quickly,* whereas the movements in his experiments with the plank were executed slowly. In order to exclude with certainty labyrinthine reflexes it would have been more conclusive if he had been able to demonstrate the absence of eye-movements in moving the plank quickly. The possibility remains that in his case reflexes of the semi-circular canals were present, which were failing in the experiments with the plank, because the movements were then executed too slowly to surpass the threshold of eliciting these reflexes.

Some months before we read the communication of SCHUSTER we were able to examine a patient with vertical tonic neck-reflexes on the eyes in the clinic of Professor WINKLER at Utrecht. Remarkably enough in this case we had eliminated tonic labyrinthine reflexes in exactly the same way as described above. As it could be settled that in this patient, in moving the head slowly, the reflexes were also of a tonic character, it was evident that here tonic reflexes on the eyes were present.

Here follows the history of this patient (boy of 8 years, admitted to the clinic of Professor WINKLER from February 15th—July 1st, 1921).

Previous history (February 15th 1921).

Birth was normal, he talked and walked at normal age, but was incontinent off and on. When three years old he had pneumonia; at six he went to school, learning normal.

Three months before admission his walk became disturbed, he stumbled over everything, and sometimes fell to the ground. He preferred to sit in a chair. Vomiting, independent of eating, developed and he suffered from increasing head-ache. The patient was right-handed, had never suffered from epilepsy, never been unconscious. In November 1920 he complained of diplopia.

Family history: mother had tuberculosis, died of influenza. No further particularities.

Neurologic examination:

The right papilla was pale, not completely circumscribed, the excavation was just indicated. In the reversed image a vessel going upward was heavily twisted. The left papilla was not circumscribed, here and there the heavily twisted vessels were covered with exsudate.

Patient was lying quietly, no straight position. Scent intact on both sides, pupils round, of equal size and middle wide, reacted to light and convergence. The visual field was not limited. The trigeminus was bilaterally intact as far as this was ascertainable by examination of the sensation for pain, touch and of the motoric function. The cornea reflexes were bilaterally vivid.

On showing the teeth the right side was not innervated so well as the left side, just as in mimical movements. The palatum moved symmetrically, the pharynx reflex was

positive, the taste was intact. The tongue protruded in midline, the arm movements were strong, perhaps more pronounced on the right. Biceps and triceps reflexes were present.

Dysdiadochokinesis bilaterally present, more pronounced on the left than on the right. However, in children this is a physiologic phenomenon.

The abdominal reflexes were absent on both sides, the cremaster, the tendon (supra- and infrapatellar) reflexes were bilaterally present. There existed a BABINSKI-reflex at both sides. The skin was difficult to be lefted.

When walking the patient showed a strong ataxia with a tendency to fall backward, to the right and to the left. When moving from the dorsal position, arms crossed, the typical asynergia of BABINSKI developed. The same took place when the patient bent backward. In sitting position the patient showed a tendency to fall to the right.

The X ray of the skull showed an enlargement of the sella turcica; besides, the deepened impressiones digitatae direct the attention to an increased chronic intra-cranial pressure. Urine: no abnormalities. After a stay of some months in the clinic the condition seemed to be much improved.

On June 13th, 1921, we find the following note: patient looks well, feels comfortable, no vomiting, no head-ache. He has become corpulent and has got a habitus feminus.

Walking, though still uncertain, had improved. Ophthalmoscopic examination proved that the turbicity of the left papil had disappeared (atrophying papilla-oedema). The patellar reflex had become plantar on the left, on the right it still showed the BABINSKI-type. The asynergia of BABINSKI when rising and bending backward had diminished but was still present.

However, the X ray made on June 29th 1921, revealed a progressive course (impressiones more pronounced, sella turcica more dilated).

The ear-drums were normal, slight adenoid with enlarged tonsils.

Acoustic examination: no abnormalities on either side, the whispering voice was heard at normal distance, the tuning fork test was also completely normal for both sides.

Vestibular examination: No nystagmus when in rest. When looking to the right or to the left the eyes remained fixed in the middle position, just as when patient was asked to follow a finger which was moved horizontally before his eyes. Convergence and looking upward intact (here divergent position of the eyes developed with vertical nystagmoid movements. The upper eye-lid was moved well upward. Looking downward intact, a convergent position developed.

No spontaneous past pointing in both shoulder-joints.

The caloric examination of both labyrinths after syringeing the ears with cold water proved that no nystagmus developed in the optimal position of the horizontal semi-circular canals. Reactive past pointing after syringeing the right ear was typical, syringeing the left ear elicited a reactive past pointing only on the right side.

When the patient was placed on the revolving-chair no horizontal after-nystagmus was present neither after turning to the right nor after turning to the left. A strong reactive past pointing in both shoulder-joints was present.

Neither a vertical nor a rotatory nystagmus could be elicited after turning to the right and to the left. However, the vertical optocinetic nystagmus (examined with the cylinder of BÁRÁNY) was quite normal in both directions.

Tonic neck-reflexes on the eyes were noted on various dates, e.g. Febr. 15th, March 1st and 5th, and could be demonstrated to Professor MAGNUS and Professor BÁRÁNY who happened to be at Utrecht:

a. Slow bending of the head to the chest (so that no reflexes of the semi-circular canal could be elicited): both eyes went upward and remained in this position as long as the head was inclined on the chest.

b. Very slow bending of the head backward: the eyes turned downward and were kept in this position as long as the head remained fixed.

This phenomenon also developed when the spectacles of BARTELS (20 dioptria) were placed before the eyes to prevent fixation; the patient being asked to look forward quietly.

To establish if tonic neck or tonic labyrinthine reflexes were present, the following test was performed.

The patient was placed on a stretcher and firmly fixed to it. Care was taken that, when moving the stretcher, the position of the head in relation to the trunk did not change.

Not a single alteration of position of the eyes was noted if, by very slow movements, the stretcher was so turned that the patient had the following successive positions: head downward, head upward, in dorsal, in abdominal and in the interjacent positions.

One could now conclude that the positions of the eyes, found in the various positions obtained by turning the head, were due to tonic neck-reflexes and not to tonic labyrinthine reflexes.

The patient was discharged and, contrary to all expectation, survived.

It was possible to re-examine this patient in November 1939. He was very slow in intellect, but felt completely healthy. He had had a bilateral otitis. Neurological examination revealed no abnormalities except a slight right-sided facialis paresis.

Vestibular function seemed to be the same: now, however, it appeared that in the so-called "subcortical" form of the optocinetic nystagmus, the horizontal nystagmus to the right and to the left was completely absent and the vertical nystagmus up- and downward nearly so. In the "cortical" form however the vertical nystagmus up- and downward could be elicited easily, just as in 1921. The horizontal nystagmus to the right and to the left was, in this cortical form also absent [1]).

The tilting-reactions of RADEMAKER and GARCIN could be elicited both round the bitemporal and the longitudinal axis.

The different forms of nystagmus after rotation (horizontal, vertical and rotatory) were still absent.

Caloric stimulation did not cause a horizontal or rotatory nystagmus. Contrary to our findings in 1921, the reactive past pointing was intact on both sides. It was remarkable that in the different vestibular stimulations, dizziness was completely absent.

It was now impossible to elicit the tonic neck-reflexes on the eyes.

Our knowledge of the anatomical base, on which these vertical tonic neck-reflexes on the eyes develop, is still limited. In the case of BERNHARDT and SIMONS the typical syndrome of an encephalitis lethargica was found [2]); SCHUSTER, in his case, thought that a circumscribed focus was

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[1]) This remarkable difference in the examination of the "subcortical" and "cortical" form of the optocinetic nystagmus will be discussed in another communication.

[2]) The description of the autopsy of the patient of BERNHARDT and SIMONS was as follows:

Die histologische Untersuchung (Demonstration) ergab, dasz die pathologischen Veränderungen vorwiegend auf das zentrale Höhlengrau, die Haube, die Substantia reticularis der Brücke beschränkt waren und zur Medulla oblongata hin verschwanden. Es

present because only the tract for the "Spähbewegung", i.e. the tract between the cortical and subcortical centra, of the congregate movements respectively the eye nuclei, was blockaded and the junction of the eye-nuclei with the optical sphere, the vestibular nuclei and the sensible tracts from the upper spinal cow was still intact.

Our case was only clinically examined. However, it is likely that our patient suffered also from a circumscribed process in that part of the central nervous system.

It was recorded by VOSS (l.c.) that these reflexes could also be seen in neonati and premature infants.

C. *Rotatory eye-positions caused by tonic neck-reflexes.*

As mentioned above, also in normal individuals, reflectory rotatory eye-positions may be present. However, rotatory eye-movements cannot be executed voluntarily and so cannot have a disturbing influence when rotatory tonic labyrinthine or neck reflexes on the eyes are elicited.

In literature but a few data are found about these reflexes. This must be due to the technical difficulties which are connected with the determination of the rotatory eye-positions. VERSTEEGH and DE KLEYN [1]) described a method to examine the compensatory labyrinthine counter-rolling of the eyes in human beings. A membrane on which a cross was marked, was constructed on the eye. The eye was photographed with the patient in the sitting and in the two lateral positions. The counter-rolling can be determined directly in photographing, together with the membrane, a coordinationsystem fixed to the eye.

In order to exclude the development of neck-reflexes on the eyes care was taken that the position of the head in relation to the trunk should remain the same in the different positions of the patient.

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fand sich *stärkste perivaskuläre Infiltration mit Blutungen in den Hisschen Raum.* Die Infiltrate setzten sich nicht auf die kleinsten Gefäsze fort, sondern hörten meist an den Vorkapillaren auf. Kapillaren selbst meist trotzend mit Blut gefüllt. *Ferner diffuse Gewebsinfiltration, die sich elektiv auf die graue Substanz beschränkte.* Die Infiltratzellen bestanden vorwiegend aus *Lymphozyten* und sogenannten *Polyblasten,* spärlich Plasmazellen und ganz vereinzelt polynukleären Leukocyten. Auch die *Neuronophagie* (Neurocytophagie) fand sich in charakteristischer Weise, wenn auch im ganzen nicht gerade häufig: das An- und Eindringen von Rundzellen und Polyblasten in die Ganglienzellen, kurz, genau dieselben Bilder, wie sie *Economo* beschreibt und abbildet. Auch die Veränderungen an den Nervenzellen, die PIERRE MARIE und TRETIAKOFF bei zwei Fällen von Encephalitis lethargica erwähnen, das Herandrängen des Kerns an die Peripherie, hyaline Degeneration desselben, schlieszlich Restieren eines einfachen Bläschens mit Körnchen darin, also *akuter Zellzerfall* kamen zur Beobachtung. Gewöhnlich traten Gefäszinfiltration, diffuse Infiltration und Neuronophagie zusammen auf, aber, worauf *Economo* besonderen Wert legt, es fanden sich auch in der grauen Substanz isoliert kleinste neurophage Herden um eine Ganglienzelle.

[1]) KLEYN, A. DE and VERSTEEGH, C., Acta Oto-Laryng., **6**, 170 (1923); Jnl. Laryng. and Otology, **39**, 686 (1924).

Our following case shows the necessity of these precautions [1]). v. W., female, 17 years.

This patient had become deaf-mute after scarlet fever when 4 years old. The ear-drums were absent at both sides. There was complete deafness on both ears. It was impossible to stimulate the labyrinths calorically or to elicit one of the forms of nystagmus (horizontal, vertical or rotatory) after turning to the left and to the right.

Examination of the compensatory labyrinthine eye-positions (the eyes being photographed with the patient in dorsal and in both lateral positions) proved that these were absent. However, after inclining the head of the patient (in sitting position) to one of the shoulders, a distinct counter-rolling of $2\frac{1}{2}°$ developed, especially when inclining it to the left shoulder.

In literature only a few examples of these rotatory tonic neck-reflexes are recorded. This is due to the fact that the compensatoric eye-positions were but seldom clinically registrated and if registrated *those* methods were used, by which the labyrinthine and neck-reflexes are determined together.

It is evident that a more simple method must be found for the clinical determination of the rotatory eye-positions [2]). It is very probable that rotatory tonic neck-reflexes might be found in a large number of individuals. For this purpose it would only be necessary to make five photographs: eye-position in sitting position, eye-position in both lateral positions (head symmetrical in relation to the trunk: labyrinthine reflexes); eye-positions in turning the head to the right and to the left shoulder (labyrinthine *and* neck reflexes).

In neonati and premature children VOSS [3]) described tonic rotatory neck-reflexes on the eyes; FISCHER [4]) observed these in a patient whose labyrinths could not be stimulated.

In rare cases (observation of one of us, STENVERS 1918 [5])) it was possible to elicit tonic reflexes on the eyes not by changing the position of the head but by changing the position of the pelvis in relation to the trunk.

Another group of neck-reflexes on the eyes probably comprises those reflexes, developing during the turning of the head in relation to the trunk or during the turning of the trunk in relation to the head. This is a nystagmus of only a short duration which can be noted in some cases *during the movement of the head or trunk,* if the position was mutually changed and is not followed by a remaining change of position of the eyes. Such cases

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1) KLEYN, A. DE and VERSTEEGH, C., Jnl. Laryng. and Otology, **39**, 686 (1924).

2) The above mentioned methods are also taking up much time.

3) VOSS, O., Folio Oto-Laryng., II, **24**, 16 (1925).

4) FISCHER, M. H., v. Graefe's Arch., **118**, 633 (1927); Acta Oto-Laryng., 8, 495 (1925); Regulationsfunktion des menschlichen Labyrinthes (Bergmann, München, p. 92 (1928)).

5) STENVERS, H. W., Arch. néerl. de phys. de l'homme et des animaux 2, 669 (1918).

are described by FRENZEL [1]) and GÜTTICH [2]). GÜTTICH could demonstrate that after an anesthesia of the cervical roots it was more difficult to elicit a nystagmus and that then this nystagmus was but a slight one.

The nystagmus of the eyes obtained when turning the pelvis, are described by GRAHE [3]).

Summary.

Taking literature as a basis and recording own cases, different forms of tonic neck-reflexes on the eyes are discussed. Besides the way in which these reflexes can be examined is mentioned.

Under normal circumstances, horizontal and vertical eye-positions, caused by tonic neck-reflexes are never seen in healthy individuals except in neonati. They are up to now only met with in serious affections of the central nervous system (tumours, encephalitis), whereas rotatory eye-positions caused by tonic neck-reflexes can also be elicited in healthy individuals.

The necessity of better clinical methods for examining these latter reflexes is emphasized.

Some cases of vertical eye-positions and the only case of horizontal eye-positions in adults, caused by tonic neck-reflexes, are extensively recorded.

Another group of neck-reflexes on the eyes includes the temporary nystagmus developing in some patients *during* the turning of the head in relation to the trunk or inverse (FRENZEL, GÜTTICH).

In some cases tonic reflexes and reflexes responding to movements of the eyes are obtained by changing the position of the pelvis in relation to the trunk. All that has been communicated about the neck-reflexes on the eyes also appears to hold good for these reflexes.

[1]) FRENZEL, H., Ztschr. f. Hals-, Nasen- und Ohrenh., **21**, 177 (1928), Passow-Schaefer, **28**, 305 (1931).

[2]) GÜTTICH, Verh. Ges. D. Hals-, Nasen- und Ohrenärzte, 209 (1933).

[3]) GRAHE, K., Ztschr. f. Hals-, Nasen- und Ohrenh., **13**, 613 (1926).

Mathematics. — *Folgen und Reihen in bewerteten Körpern.* (Erste Mitteilung.) II. By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of February 22, 1941.)

§ 5. *Konvergenzkriterien.*

Wir haben schon für eine Folge von Elementen $\{x_n\}$ aus einem bewerteten Körper K das allgemeine Konvergenzkriterium bewiesen und daraus ergab sich das Konvergenzkriterium für Reihen. Es gibt noch einige weniger allgemeine Eigenschaften und Konvergenzkriterien für unendliche Reihen in bewerteten Körpern.

1. Multipliziert man alle Glieder eine Reihe $\sum_{n=0}^{\infty} u_n$ mit einem Element a von K, so bleibt die neue Reihe konvergent oder divergent zugleich mit der ersten Reihe. War die ursprüngliche Reihe konvergent und $\{s_n\} \to S$, so nähert ist die neue Summe $a \cdot S$.

2. Wenn $\sum_{n=0}^{\infty} u_n = S$ und $\sum_{n=0}^{\infty} v_n = S'$, so ist $\sum_{n=0}^{\infty} (u_n + v) = S + S'$.

3. Eine Reihe ist konvergent, wenn sie absolut konvergiert. Wir benutzen dieses Kriterium für den Fall, dasz gegeben sei, dasz die Bewertungen der Glieder einer unendlichen Reihe den entsprechenden Gliedern einer konvergenten Reihe mit reellen positiven Gliedern höchstens gleich sind, dann konvergiert $\sum_{n=0}^{\infty} |u_n|$, folglich auch $\sum_{n=0}^{\infty} u_n$.

4. Wenn für alle hinreichend groszen n der Ausdruck $\sqrt[n]{|u_n|}$ kleiner als eine positive Zahl $k < 1$ bleibt, so ist die Reihe $\sum_{n=0}^{\infty} u_n$ konvergent; wenn aber $\sqrt[n]{|u_n|}$ für alle hinreichend groszen $n \geqq 1$ bleibt, so divergiert $\sum_{n=0}^{\infty} u_n$.

Ist nämlich

$$\sqrt[n]{|u_n|} < k < 1,$$

so ist

$$|u_n| < k^n$$

und die Glieder der Reihe $\sum_{n=0}^{\infty} u_n$ sind für alle hinreichend groszen n kleiner

als die entsprechenden positiven Glieder einer geometrischen Reihe. Also konvergiert $\sum_{n=0}^{\infty}|u_n|$ und folglich $\sum_{n=0}^{\infty}u_n$.

Ist für hinreichend groszes n

$$\sqrt[n]{|u_n|} \geqq 1, \text{ dann ist also } |u_n| \geqq 1.$$

Nun ist für die Konvergenz jeder Reihe notwendig, dasz $\lim_{n\to\infty} u_n = 0$, also $\lim_{n\to\infty}|u_n| = 0$; wenn also für alle hinreichend groszen n

$$|u_n| \geqq 1,$$

so wird diese notwendige Bedingung nicht erfüllt, also divergiert $\sum_{n=0}^{\infty}u_n$.

5. Wenn für alle hinreichend groszen n der Ausdruck:

$$\frac{|u_{n+1}|}{|u_n|} < k < 1$$

bleibt, so konvergiert $\sum_{n=0}^{\infty}u_n$, aber bleibt dieses Verhältnis mindestens 1, so divergiert $\sum_{n=0}^{\infty}u_n$.

Bleibt nämlich für hinreichend groszes n:

$$|u_{n+1}| < k \cdot u_n,$$

so ist auch

$$|u_{n+2}| < k^2 \cdot u_n, \text{ u.s.w,}$$

also

$$|n_{n+1}| + |u_{n+2}| + \ldots < (k + k^2 + \ldots) \cdot |u_n|$$

d.h. die Reihe der Bewertungen konvergiert, also ebenfalls die Reihe $\sum_{n=0}^{\infty}u_n$ für $k < 1$. Hinsichtlich der Divergenz:

$$\frac{|u_{n+1}|}{|u_n|} \geqq 1$$

für alle hinreichend groszen n bedeutet

$$|u_{n+1}| \geqq |u_n|$$

d.h. $\lim_{n\to\infty} u_n \neq 0$, also divergiert $\sum_{n=0}^{\infty} u_n$.

6. Wenn die beiden Ausdrücke $\sqrt[n]{|u_n|}$ und $\frac{|u_{n+1}|}{|u_n|}$ einen Grenzwert für $n \to \infty$ besitzen, so sind diese Werte einander gleich.

Betrachten wir die Potenzreihe:

$$u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n + \dots$$

Ist nun:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|u_n x^n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|u_n|} \cdot |x| = k \cdot |x|,$$

so konvergiert die Potenzreihe für $k|x| < 1$, also für $|x| < \frac{1}{k}$, und zwar absolut und divergiert für $|x| > \frac{1}{k}$.

Ist

$$\overline{\lim_{n\to\infty}} \frac{|u_{n+1} x^{n+1}|}{|u_n x^n|} = \overline{\lim_{n\to\infty}} \frac{|u_{n+1}|}{|u_n|} \cdot |x| = k' \cdot |x|$$

so konvergiert die Potenzreihe für $|x| < \frac{1}{k'}$ und divergiert für $|x| > \frac{1}{k'}$; weil die beiden Bedingungen einander nicht ausschlieszen dürfen, musz $k = k'$ sein.

7. Wenn für alle hinreichend groszen n der Ausdruck

$$\frac{\log \frac{1}{|u_n|}}{\log n} > k > 1$$

bleibt, so ist die Reihe $\sum_{n=0}^{\infty} u_n$ konvergent. Aus $\frac{\log \frac{1}{|u_n|}}{\log n} k$ folgt nämlich $\log \frac{1}{|u_n|} > k \cdot \log n$ und also $|u_n| < n^{-k}$. Die Reihe $\sum_{n=0}^{\infty} n^{-k}$ konvergiert für $k > 1$, also $\sum_{n=0}^{\infty} u_n$ ebenfalls für $k > 1$. Bleibt aber für alle hinreichend groszen n

$$\frac{\log \frac{1}{|u_n|}}{\log n} \leqq 1, \text{ so ist offenbar } |u_n| \geqq \frac{1}{n},$$

und also divergiert die Reihe der Bewertungen. Im archimedischen Falle kann man also für die Reihe $\sum_{n=0}^{\infty} u_n$ nicht auf Konvergenz schlieszen. Im nicht-archimedischen Fall divergiert $\sum_{n=0}^{\infty} u_n$ gewisz, denn die absolute Konvergenz ist für nicht-archimedische Bewertungen eine Notwendigkeit.

§ 6. *Multiplikation von Reihen.*

Nehmen wir an:

$$u_0 + u_1 + u_2 + \dots + u_n + \dots \tag{1}$$

und

$$v_0 + v_1 + v_2 + \dots v_n + \dots \tag{2}$$

seien zwei konvergente Reihen mit Gliedern aus einem bewerteten Körper. Durch Multiplikation eines Gliedes der ersten Reihe mit einem Gliede der zweiten Reihe und durch Zusammenfügung aller Produkte $u_i v_j$, wofür $i + j$ gleich ist, erhalten wir eine neue Reihe:

$$u_0 v_0 + (u_0 v_1 + u_1 v_0) + \dots + (u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0) + \dots \tag{3}$$

Es ist zu beweisen, dasz diese Reihe konvergiert und zwar zum Produkt der gegebenen Reihen, wenn wir voraussetzen, dasz eine der beiden gegebenen Reihen absolut konvergiert. Setzen wir also voraus, dasz (1) absolut konvergiert und dasz (2) konvergiert. Es sei w_n das allgemeine Glied der Reihe (3)

$$w_n = u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0.$$

Wir beweisen, dasz die Differenz

$$w_0 + w_1 + \dots + w_{2n} - (u_0 + u_1 + \dots + u_n)(v_0 + v_1 + \dots + v_n)$$

und ebenfalls

$$w_0 + w_1 + w_2 + \dots + w_{2n+1} - (u_0 + \dots + u_{n+1})(v_0 + v_1 + \dots + v_{n+1})$$

den Grenzwert Null hat, wenn $n \to \infty$.

Der Beweis ist für beide Fälle übereinstimmend; wir geben ihm für die erste Differenz, die wir also ordnen:

$$\begin{aligned}\Delta = u_0 (v_{n+1} + \dots + v_{2n}) + u_1 (v_{n+1} + \dots + v_{2n-1}) + \dots + u_{n-1} v_{n+1} + \\ + u_{n+1} (v_0 + \dots + v_{n-1}) + u_{n-2} (v_0 + \dots + v_{n-2}) + \dots + u_{2n} v_0.\end{aligned}$$

Die Reihe (1) konvergiert absolut, also ist die Summe

$$|u_0| + |u_1| + \dots + |u_n| < A$$

für alle n, worin A eine reelle positive Zahl ist. Wegen der Konvergenz der Reihe (2) ist für alle n:

$$|v_0 + v_1 + \dots + v_n| < B,$$

worin B eine reelle positive Zahl ist. Es sei nun $\varepsilon > 0$ eine bestimmte reelle Zahl, so ist eine Zahl m festzustellen, so dasz

$$|u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}| < \frac{\varepsilon}{A + B} \text{ für alle } p \text{ und } n \geqq m$$

und

$$|v_{n+1}+v_{n+2}+\ldots+v_{n+p}|<\frac{\varepsilon}{A+B} \text{ für alle } p \text{ und } n \geqq m.$$

Hat man hieraus n bestimmt, so ist für $|\triangle|$ eine Schranke anzugeben:

$$|\triangle| \leqq |u_0(v_{n+1}+\ldots+v_{2n})|+|u_1(v_{n+1}+\ldots+v_{2n-1})|+\ldots+|u_{2n}v_0|$$

$$|\triangle| \leqq |u_0|\cdot\frac{\varepsilon}{A+B}+|u_1|\frac{\varepsilon}{A+B}+\ldots+$$

$$+|u_{n-1}|\cdot\frac{\varepsilon}{A+B}+|u_{n+1}|B+|u_{n+2}|B+\ldots+|u_{2n}|B$$

$$|\triangle| \leqq \frac{\varepsilon}{A+B}(|u_0|+|u_1|+\ldots+|u_{n-1}|)+$$

$$+B(|u_{n+1}|+|u_{n+2}|+\ldots+|u_{2n}|)$$

$$|\triangle| \leqq \frac{\varepsilon A}{A+B}+\frac{\varepsilon B}{A+B} \text{ also } |\triangle| \leqq \varepsilon.$$

Hieraus ergibt sich die Konvergenz der Produktreihe.

§ 7. *Unendliche Reihen in einer Veränderlichen.*

Betrachten wir eine unendliche Reihe

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \tag{1}$$

worin die a Elemente eines vollständigen bewerteten Körpers sind und untersuchen wir die Konvergenz der Reihe (1). Setzen wir $|a_i|=A_i$, und sind also die A_i reelle Zahlen. Die Reihe

$$A_0+A_1X+A_2X^2+\ldots+A_nX^n+\ldots \tag{2}$$

für welche X nur reeller Gröszen fähig ist, konvergiere für $X<R$ und divergiere für $X>R$. Betrachten wir nun (1); die Reihe konvergiert absolut für $|x|<R$. Es soll noch gezeigt werden, dasz (1) für $|x|>R$ divergiert; dafür benutzen wir folgenden Satz: „Konvergiert die Reihe (1) für $x=x_0$, so konvergiert die Reihe absolut für alle $|x|<|x_0|$.

Es sei nämlich M eine reelle positive Zahl, gröszer als alle Glieder der Reihe

$$\sum_{n=0}^{\infty}|a_nx_0^n|,$$

so ist

$$|a_n|\cdot|x_0|^n<M.$$

Nun ist

$$|a_n x^n| = |a_n|\,|x_0^n| \cdot \left(\frac{|x|}{|x_0|}\right)^n < M\left(\frac{|x|}{|x_0|}\right)^n,$$

also konvergiert $\sum_{n=0}^{\infty} |a_n x^n|$ für alle $|x| < |x_0|$, aber für diese x konvergiert die Reihe (1) absolut.

Es kann also nicht $|x_0| > R$ sein, sonst könnte R nicht die obere Grenze der $|x|$ sein, für welche

$$\sum_{n=0}^{\infty} |a_n|\,|x^n|$$

konvergiert.

Man erhält also: Die Reihe (1) ist absolut konvergent für alle x mit $|x| < R$ und divergiert für $|x| > R$, wenn R den Konvergenzradius der Reihe

$$\sum_{n=0}^{\infty} |a_n|\, X^n$$

darstellt.

Stetigkeit einer Potenzreihe. Es sei $f(x) = a_0 + a_1 x + a_n x^n + \ldots$ (1) die Reihe, welche für $|x| < R$ konvergiert, und es sei auszerdem R' eine positive Zahl mit $R' < R$. Wir zeigen nun, dasz die Reihe (1) gleichmäszig konvergent ist für $|x| \leqq R'$.

Für $|x| \leqq R'$ ist $|a_n x^n|$ höchstens dem entsprechenden Gliede der konvergenten Reihe

$$|a_0| + |a_1|\, R' + |a_2|\, R'^2 + \ldots$$

gleich, also

$$\left|\sum_{\nu=n+1}^{n+p} a_\nu x^\nu\right| \leqq \sum_{\nu=n+1}^{n+p} |a_\nu| \cdot |x^\nu| \leqq \sum_{\nu=n+1}^{n+p} |a_\nu|\, R'^\nu.$$

Weil R der Konvergenzradius der Reihe $\sum_{\nu=0}^{\infty} |a_\nu|\,|x|^\nu$ ist, kann man also jeder positiven Zahl $\varepsilon > 0$ ein n_0 zuordnen, so dasz für alle $n > n_0$ und $p \geqq 1$

$$|a_{n+1}| \,.\, R'^{\,n+1} + |a_{n+2}| \,.\, R'^{\,n+2} + \ldots + |a_{n+p}|\, R'^{\,n+p} < \varepsilon.$$

Es ist dann für alle $|x| < R'$:

$$|a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots + a_{n+p} x^{n+p}| < \varepsilon$$

also konvergiert die Reihe (1) gleichmäszig für $|x| < R'$, und sie stellt, wie früher bewiesen ist, eine stetige Funktion dar für $|x| < R'$.

Da man R' willkürlich nahe an R wählen kann, erhält man folgenden

Satz: Die Potenzreihe (1) stellt für $|x| < R$ eine stetige Funktion dar und konvergiert für dieses x gleichmäszig.

R wird bestimmt mit der Folge:

$$|a_1|, \sqrt{|a_2|}, \sqrt[3]{|a_3|}, \ldots, \sqrt[n]{|a_n|}, \ldots$$

Es sei $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} = \omega$.

1⁰. Ist $\omega = 0$, so musz die Reihe für jedes x von K konvergieren. Es sei nämlich x_1 ein beliebiges Element von K, also

$$\frac{1}{2|x_1|} > 0$$

dann ist der Bedeutung gemäsz von ω für alle hinreichend groszen n:

$$\sqrt[n]{|a_n|} < \frac{1}{2|x_1|}, \text{ also } |a_n| < \frac{1}{2^n \cdot |x_1|^n}.$$

Hieraus folgt

$$|a_n x_1^n| < \frac{1}{2^n}$$

woraus man die Konvergenz der Reihe $\sum\limits_{n=0}^{\infty} |a_n x_1^n|$, also der Reihe (1) für jedes x_1 von K schlieszt.

2⁰. Konvergiert umgekehrt (1) für $x = x_1$, so ist die Folge $\{|a_n x_1^n|\}$ beschränkt, und folglich ebenfalls die Folge $\{\sqrt[n]{|a_n x_1^n|}\}$. Ist also

$$\sqrt[n]{|a_n x_1^n|} < M,$$

so folgt hieraus:

$$\sqrt[n]{|a_n|} < \frac{M}{|x_1|},$$

d.h. die Menge $\{\sqrt[n]{|a_n|}\}$ ist beschränkt.

Ist $\omega = \infty$, so ist die Folge $\{\sqrt[n]{|a_n|}\}$ unbeschränkt und die Reihe (1) kann für kein x von K konvergieren.

3⁰. Ist $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} = \omega$, so ist $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} \cdot |x| = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n x^n} = \omega |x|$, und es konvergiert die Reihe (1) für $\omega|x| < 1$, also für $|x| < \frac{1}{\omega}$ und sie divergiert für $|x| > \frac{1}{\omega}$.

Beispiele: Die Reihe $\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ konvergiert für alle bewerteten Körper für $|x|<1$, denn $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\left|\frac{\pm 1}{n}\right|}=1$.

Die Reihen

$$\left.\begin{array}{l} 1+\frac{x}{1}+\frac{x^2}{2!}+\ldots \\[2ex] 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots \\[2ex] x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots \end{array}\right\} \tag{3}$$

konvergieren im Körper der reellen Zahlen für alle x. Für eine p-adische Bewertung ist der Konvergenzradius

$$R=p^{-\frac{1}{p-1}}.$$

Dafür benutzen wir folgende Tatsache:

Ist $m=a_0, a_1\ldots a_n$ die Darstellung einer beliebigen gewöhnlichen ganzen Zahl m in der reduzierten Form, so ist das Produkt $m!$ genau durch

$$p^{\frac{m-s_m}{p-1}}$$

teilbar, wenn $s_m=a_0+a_1+\ldots+a_r$ die p-adische Ziffersumme von m bedeutet. Es ist

$$1<s_m<(p-1)^p\log(m+1), \text{ also } \lim_{m\to\infty}\frac{s_m}{m}=0.$$

Zum Beweise der obigen Formel zeigen wir, dasz die Ordnungszahl eines beliebigen ganzen rationalen n

$$\nu=\frac{s_{n-1}-s_n+1}{p-1}$$

ist; ist nämlich

$$n=0,000a_\nu\, a_{\nu+1}\ldots a_r \quad (p)$$

die Darstellung dieser Zahl n in der reduzierten Form, so ist

$$n-1=p-1, p-1\; p-1\; p-1\ldots p-1\; a-1\; a_\nu-1\; a_{\nu+1}\ldots a_r \quad (p)$$

und aus den beiden Ziffersummen

$$s_n=a_\nu+a_{\nu+1}+\ldots+a_r$$
$$s_{n-1}=\nu(p-1)+a_\nu-1+a_{\nu+1}+\ldots+a_r$$

erhält man in der Tat durch Subtraktion die obige Gleichung für die Ordnungszahl ν, welche auch für $n=1$ gilt, da ja die Ziffersumme s_0 von Null gleich Null ist.

Aus dieser Formel folgt sofort für die Ordnungszahl μ_m des Produktes

$$1 \cdot 2 \cdot 3 \cdots m = m!$$

die Gleichung:

$$\mu_m = \frac{1}{p-1} \sum_{n=1}^{m} (s_{n-1} - s_n + 1) = \frac{m - s_m}{p-1} \quad \text{w. z. b. w.}$$

Also ist

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{1}{n!}\right|} = \overline{\lim_{n\to\infty}} \left(p^{\frac{1}{p-1} - \frac{s_n}{(p-1)n}}\right) = p^{\frac{1}{p-1}}.$$

Die Reihen (3) konvergieren für alle x, für welche $|x| < p^{-\frac{1}{p-1}}$, und sie divergieren für $|x| > p^{-\frac{1}{p-1}}$.

Aus diesem Satz folgt:

Die Reihe

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots (p) \text{ divergiert}$$

und

$$1 + \frac{p}{1} + \frac{p^2}{2!} + \frac{p^3}{3!} + \dots (p) \text{ konvergiert.}$$

Ableitung einer Potenzreihe. Besitzt die Potenzreihe (1) den Konvergenzradius R, so ist auch die Reihe der Ableitungen

$$a_1 + 2a_2 x + \dots + n a_n x^{n-1} + \dots$$

konvergent für $|x| < R$ und divergent für $|x| > R$ und stellt für jene x die Funktion $f'(x)$ dar. Das erste folgt sofort aus der Tatsache:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|n a_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}.$$

Es sei die Summe der Reihe der ersten Ableitungen $f_1(x)$, so ist diese Funktion für $|x| < R$ eine stetige Funktion, und die Reihe konvergiert gleichmäszig für $|x| \leqq R'$ $(R' < R)$. Wegen des zweiten Satzes der gleichmäszigen Konvergenz für analytische Funktionen (S. 295) stellt dann $f_1(x)$ die Funktion $f'(x)$ dar und wir erhalten $f'(x)$ durch gliedweise Differenzierung. Ebenso kann man durch Wiederholung die Ableitung $f''(x)$ erhalten, ebenfalls eine Potenzreihe, welche für $|x| < R$ konvergiert und für $|x| > R$ divergiert. Die Funktion $f(x)$ gestattet also für $|x| < R$ eine

unbeschränkte Anzahl Ableitungen, welche man erhält durch gliedweise Differenzierung:

$$f^{(n)}(x) = 1 \cdot 2 \cdots n\, a_n + 2 \cdot 3 \cdots n\,(n+1)\, a_{n+1}\, x + \ldots$$

Für $x = 0$ ist also

$$a_0 = f(0);\ a_1 = f'(0);\ a_2 = \frac{1}{2!} \cdot f''(0);\ \ldots;\ a_n = \frac{f^{(n)}(0)}{n!},$$

und somit

$$f(x) = f(0) + \frac{x}{1} f'(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

Setzt man voraus, dasz (1) für $|x| < R$ konvergiert, und dasz $|x_0| + |h| < R$, so erhält man:

$$\begin{aligned}
f(x_0 + h) &= a_0 + a_1(x_0 + h) + a_2(x + h)^2 + \ldots + a_n(x_0 + h)^n + \ldots \\
&= a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n + \ldots \\
&\quad + a_1 h + 2 a_2 x_0 h + \ldots + n\, a_n x_0^{n-1} h + \ldots \\
&\quad + a_2 h^2 + \ldots
\end{aligned}$$

Diese Reihe ist absolut konvergent; es ist ja:

$$\begin{aligned}
&|a_0| + |a_1|\,|x_0| + |a_2|\,|x_0^2| + \ldots \\
&\quad + |a_1|\,|h| + |2a_2|\,|x_0 h| + \ldots = \\
&|a_0| + |a_1|(|x_0| + |h|) + |a_2|(|x_0| + |h|)^2 + \ldots
\end{aligned}$$

und wegen $|x_0| + |h| < R$ konvergiert also die letzte Reihe und folglich auch

$$\sum_{n=0}^{\infty} a_n (x_0 + h)^n.$$

Auszerdem leuchtet ein, dasz

$$f(x_0 + h) = f(x_0) + \frac{h}{1} f'(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + \ldots \quad (|x_0| + |h| < R).$$

§ 8. *Majoranten.*

Es sei

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \tag{1}$$

eine Potenzreihe mit Koeffizienten aus einem vollständigen bewerteten Körper K und

$$\varphi(X) = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \ldots + \alpha_n X^n + \ldots$$

eine Potenzreihe, worin α_i nicht negative reelle Zahlen sind.

Man nennt $\varphi(X)$ eine Majorante von $f(x)$, also

$$f(x) \ll \varphi(X),$$

wenn $|a_n| \leqq \alpha_n$ für alle n.[4])

Ist

$$P_n(a_0, \ldots, a_n)$$

ein Polynom der a_i, in welchem ausschlieszlich die Addition und Multiplikation auftritt, so wird, wenn man die Koeffizienten a_i aus dem Körper K durch die entsprechenden reellen α_i von $\varphi(X)$ ersetzt:

$$|P_n(a_0, a_1, \ldots, a_n)| \leqq P_n(\alpha_0, \alpha_1, \ldots, \alpha_n).$$

Ist z. B. $\varphi(X)$ eine Majorante von $f(x)$, so ist $[\varphi(X)]^2$ eine Majorante von $[f(x)]^2$.

Es sei $0 < r < R$, dann konvergiert (1) absolut für $|x| = r$. Es sei M eine obere Schranke der Glieder der Reihe

$$\sum_{n=0}^{\infty} |a_n| \cdot r^n, \text{ also } |a_n| \cdot r^n \leqq M \qquad (n = 1, 2, \ldots)$$

und folglich

$$|a_n| \leqq \frac{M}{r^n} \qquad (n = 1, 2, \ldots).$$

Hieraus ergibt sich, dasz die Reihe

$$\sum_{n=0}^{\infty} M \frac{X^n}{r^n} = M + M\frac{X}{r} + M\frac{X^2}{r^2} + \ldots = \frac{M}{1 - \frac{X}{r}} \qquad (|x| < r)$$

eine Majorante ist von $f(x)$.

Besitzt $f(x)$ kein konstantes Glied, so ist

$$\varphi(X) = \frac{M}{1 - \frac{X}{r}} - M = \frac{MX}{r - X}$$

eine Majorante von $f(x)$.

Man kann für r jede positive Zahl $< R$ wählen und es ist klar, dasz das entsprechende M mit r kleiner wird; ist aber $a_0 \neq 0$, so kann M niemals kleiner als $|a_0|$ angesetzt werden.

Ist $a_0 \neq 0$, so ist immer ein solches ϱ zu bestimmen, so dasz

$$\frac{|a_0|}{1 - \frac{X}{\varrho}}$$

---

4) Der Begriff der Majorante rührt von CAUCHY her.

eine Majorante ist von $f(x)$. Es sei nämlich $M + M\dfrac{X}{r} + \ldots \; (M > |a_0|)$ eine erste Majorante.

Ist nun

$$0 < \varrho < r \, . \, \frac{|a_0|}{M}, \text{ so ist für } n \geqq 1$$

$$|a_n|\varrho^n = |a_n|r^n \, . \left(\frac{\varrho}{r}\right)^n < M \, . \, \frac{\varrho}{r} \, . \left(\frac{\varrho}{r}\right)^{n-1},$$

also

$$|a_n| \, . \, \varrho^n < |a_0| \left(\frac{\varrho}{r}\right)^{n-1},$$

also sicher

$$|a_n|\varrho^n < |a_0| \text{ wegen } 0 < \varrho < r.$$

Aber dann ist die Reihe

$$|a_0| + |a_0| \, . \, \frac{X}{\varrho} + \ldots + |a_0| \, . \, \frac{X^n}{\varrho^n} + \ldots$$

eine Majorante von (1).

Mathematics. — *Die Lösung von Differentialgleichungen in einem bewerteten Körper.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of March 29, 1941.)

§ 1. Die Definition des bestimmten Integrals einer reellen stetigen Funktion, $y = F(x)$, der reellen Veränderlichen x zwischen den Grenzen $x = a$ und $x = b$ ist folgendermaszen: Man teile das Interval $a \leq x \leq b$ irgendwie in n Teile

$$a < x_1 < x_2 \ldots < x_{n-1} < x_n = b.$$

Man wähle in jedem Intervall $x_{\lambda-1}\, x_\lambda$ einen beliebigen Punkt ξ_λ, so dasz $x_{\lambda-1} < \xi_\lambda < x_\lambda$, und bilde die Summe

$$S_n = \sum_{\lambda=1}^{n} (x_\lambda - x_{\lambda-1}) \cdot F(\xi_\lambda).$$

Wenn bei wachsendem n alle Intervalle $x_{\lambda-1}\, x_\lambda$ unter jede Grösze herabsinken, so heiszt

$$\lim_{n \to \infty} S_n = S$$

das bestimmte Integral und wird mit

$$S = \int_a^b F(x)\, dx$$

bezeichnet. Dieser Limes ist stets vorhanden und von der Auswahl der Teilpunkte unabhängig. Diese Definition des bestimmten Integrals ist jedesmal anzuwenden, wenn es möglich ist, zwei beliebige Elemente durch einen ganz im Körper verlaufenden Weg zu verbinden.

In topologisch nicht-zusammenhängenden bewerteten Körpern ist also die obengenannte Methode nicht durchzuführen. Soll also in solchen bewerteten Körpern von Integrieren die Rede sein, so musz man es anders definieren, etwa als die Umkehrung des Differenzierens.

Als Integral der Funktion $y = f(x)$ bezeichnen wir also eine Funktion $F(x)$, so dasz $F'(x) = f(x)$. Wir werden zunächst einen Existenzbeweis solcher Funktionen $F(x)$ geben für analytische Funktionen $f(x)$.

Es möge

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

eine Potenzreihe bedeuten (mit Koeffizienten !ɐ aus einem vollständigen bewerteten Körper), welche für $|x-x_0|<r$ konvergiert. Es entsteht die Frage, ob es eine konvergente Potenzreihe

$$y=F(x)=\sum_{n=0}^{\infty} b_n(x-x_0)_n \tag{2}$$

gibt, so dasz $F'(x)=f(x)$, und die für $x=x_0$ den Wert $y=y_0$ annimmt.

Durch Differenzieren von (2) bestimmen wir die Koeffizienten und aus dem Besprochenen des Paragraphen 7 (Ableitung einer Potenzreihe) folgt, dasz die Reihe

$$F(x)=y_0+\sum_{n=1}^{\infty}\frac{1}{n}\cdot a_{n-1}(x-x_0)^n$$

die Frage befriedigt.

§ 2. Betrachten wir nun die Differentialgleichung

$$y'=\sum_{\mu,\nu} a_{\mu,\nu}(x-x_0)^{\mu}(y-y_0)^{\nu} \tag{1}$$

und setzen wir voraus, dasz die $a_{\mu,\nu}$ Elemente aus einem vollständigen bewerteten Körper sind, und dasz die Reihe (1) konvergiert für

$$|x-x_0|<r_x, \quad |y-y_0|<r_y.$$

Auszerdem seien $\bar{r}_x$ und $\bar{r}_y$ zwei positiven Zahlen, so dasz $\bar{r}_x<r_x$, $\bar{r}_y<r_y$. Es handelt sich um die Frage, ob es eine Lösung $y=F(x)$ der Differentialgleichung (1) gibt, die für $x=x_0$ den Wert $y=y_0$ annimmt. Wir setzen

$$y=y_0+\frac{c_1}{1!}(x-x_0)+\frac{c_2}{2!}(x-x_0)^2+\dots \tag{2}$$

in die Differentialgleichung ein und versuchen zunächst, ob sich die Koeffizienten dieser Reihe so bestimmen lassen, dasz sie die Differentialgleichung formal befriedigt. Der Einfachheit wegen setzen wir

$$x-x_0=\xi, \quad y-y_0=\eta;$$

dann wird

$$\frac{dy}{dx}=\frac{d\eta}{d\xi}$$

und die Differentialgleichung verwandelt sich in

$$\eta'=\sum_{\mu,\nu} a_{\mu,\nu}\xi^{\mu}\eta^{\nu} \qquad (\text{für } |\xi|<r_x;\ |\eta|<r_y) \tag{3}$$

Wir haben also die Koeffizienten der Reihe

$$\eta=\frac{c_1}{1!}\xi+\frac{c_2}{2!}\xi^2+\dots \tag{4}$$

so zu bestimmen, dasz diese Reihe die Differentialgleichung (3) formal

befriedigt. Durch Differenzieren von (4) und Koeffizientenvergleichen erhält man

$$\left.\begin{array}{l} c_1 = a_{00} \\ c_2 = \varphi_2(a_{i,l}, c_1) \\ c_3 = \varphi_3(a_{i,l}, c_1, c_2) \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ c_k = \varphi_k(a_{i,l}, c_1, c_2, \ldots, c_{k-1}) \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}\right\} i + l \leqq k-1$$

Die Koeffizienten c_k werden eindeutig aus den $a_{i,l}$ (mit $i + l \leq k - 1$) und den $c_1, \ldots, c_{k-1}$ bestimmt und zwar allein durch die Operationen der Addition und Multiplikation, z.B.

$$c_2 = a_{1,0} + a_{0,1}\, c_1, \quad c_3 = a_{0,1}\, c_2 + 2a_{2,0} + 2a_{11}\, c_1 + 2a_{0,2}\, c_1^2.$$

Die mit diesen c_k eindeutig bestimmte Reihe

$$\sum_{k=1}^{\infty} \frac{1}{k!}\, c_k\, \xi^k \tag{5}$$

befriedigt also formal die Differentialgleichung (3); es entsteht nun die Frage ihrer Konvergenz. Dafür benötigen wir der Majorantenmethode von CAUCHY. Wir denken uns anstelle der Reihe (3) eine andere Reihe

$$\sum_{\mu,\nu} A_{\mu,\nu}\, \Xi^\mu\, \mathrm{H}^\nu \tag{6}$$

deren Koeffizienten $A_{\mu,\nu}$ positive, reelle Gröszen und von der Beschaffenheit, dasz

$$|\, a_{\mu,\nu} \,| \leqq A_{\mu,\nu}$$

(Ξ und H sind nur reeller Werte fähig). Es ist also die Reihe (6) eine Majorante der Reihe (3). Wir betrachten nun die Differentialgleichung

$$\mathrm{H}' = \sum A_{\mu,\nu}\, \Xi^\mu\, \mathrm{H}^\nu \tag{7}$$

setzen darin für H die Reihe

$$\mathrm{H} = \frac{C_1}{1!}\, \Xi + \frac{C_2}{2!}\, \Xi^2 + \ldots \tag{8}$$

und bestimmen die C_k so, dasz diese Reihe der Differentialgleichung (7) formal befriedigt; dann setzen sich die C_k aus den $A_{\mu,\nu}$ offenbar in der selben Weise zusammen, wie die c_k aus den $a_{\mu,\nu}$, und da die $A_{\mu,\nu}$ reell und positiv sind, werden die C_k ebenfalls reell und positiv ausfallen und auszerdem

$$|\, c_k \,| \geqq C_k \qquad (k = 1, 2, 3, \ldots)$$

Es ist also

$$\sum_{k=1}^{\infty} \frac{c_k}{k!} \xi^k \ll \sum_{k=1}^{\infty} \frac{C_k}{k!} \Xi^k.$$

Wenn nun bekannt ist, dasz die Reihe (8) für $\Xi < a$ konvergiert, so folgt hieraus, dasz auch die Reihe (5) für $|\xi| < a$ konvergiert und zwar absolut. Ist nun M eine obere Schranke aller Glieder der Reihe

$$\sum_{\mu,\nu} |a_{\mu,\nu}| \bar{r}_x^{\mu} \bar{r}_y^{\nu},$$

so ist die Funktion

$$\sum_{\mu,\nu} M \left(\frac{\Xi}{\bar{r}_x}\right)^{\mu} \left(\frac{\mathrm{H}}{\bar{r}_y}\right)^{\nu} = \frac{M}{\left(1 - \frac{\Xi}{\bar{r}_x}\right)\left(1 - \frac{\mathrm{H}}{\bar{r}_y}\right)} \qquad (\Xi < \bar{r}_x; \mathrm{H} < \bar{r}_y)$$

eine Majorante der Reihe (3). Wir haben nun die Differentialgleichung zu betrachten:

$$\mathrm{H}' = \frac{M}{\left(1 - \frac{\Xi}{\bar{r}_x}\right)\left(1 - \frac{\mathrm{H}}{\bar{r}_y}\right)} \quad \text{(mit den Anfangsbedingungen } \Xi = 0, \mathrm{H} = 0),$$

die wir in der Form

$$\left(1 - \frac{\mathrm{H}}{\bar{r}_y}\right) d\mathrm{H} = M \frac{d\Xi}{1 - \frac{\Xi}{\bar{r}_x}}$$

schreiben. In dieser Form sind die Veränderlichen getrennt, also

$$\mathrm{H} - \frac{\mathrm{H}^2}{2\bar{r}_y} = -M\bar{r}_x \log\left(1 - \frac{\Xi}{\bar{r}_x}\right) + \text{const.}$$

Das Integral, das für $\Xi = 0$ verschwindet, ergibt sich in der Form

$$\mathrm{H} = \bar{r}_y - \bar{r}_y \sqrt{1 + \frac{2M\bar{r}_x}{\bar{r}_y} \log\left(1 - \frac{\Xi}{\bar{r}_x}\right)}$$

und dieses Integral ist durch eine Reihe von der Form

$$\mathrm{H} = \frac{C_1}{1!}\Xi + \frac{C_2}{2!}\Xi^2 + \dots$$

darstellbar. Die Konvergenzbedingungen sind

$$\|\Xi\| < \bar{r}_x \text{ und } \left\|\frac{2M\bar{r}_x}{\bar{r}_y} \log\left(1 - \frac{\Xi}{\bar{r}_x}\right)\right\| < 1$$

Die letzte Ungleichheit ist äquivalent mit

$$\| \Xi \| < \bar{r}_x \left(1 - e^{-\frac{\bar{r}_y}{2M\bar{r}_x}}\right).$$

Wegen $\bar{r}_x < r_x$ und $\bar{r}_y < r_y$ konvergiert also die Reihe (5) sicher für

$$|\xi| < \varrho, \text{ worin } \varrho = r_x \left(1 - e^{-\frac{r_y}{2Mr_x}}\right),$$

und zwar absolut. Damit ist der Existenz, zugleich die eindeutige Bestimmtheit einer Potenzreihe, welche die Frage befriedigt, bewiesen.

Es ist bemerkenswert, dasz dieses Problem nur *eine* Lösung besitzt, die in eine Potenzreihe entwickelt werden kann. Im Falle einer archimedischen Bewertung des Körpers ist diese Lösung sogar die einzige Lösung. Ist dagegen der Körper nicht-archimedisch bewertet, so existieren auch andere Lösungen als jene Potenzreihe. Es gibt nämlich in nicht-archimedisch bewerteten Körpern nicht-konstante Funktionen, deren Ableitung verschwindet.

§ 3. Etwas allgemeiner kann man nun auch das simultane System von Differentialgleichungen

$$\begin{gathered} y'_1 = f_1(x, y_1, y_2, \ldots, y_n) \\ \ldots\ldots \\ y'_n = f_n(x, y_1, y_2, \ldots, y_n) \end{gathered}$$

betrachten mit den Anfangsbedingungen $x = x_0$, $y_1 = p_1$, $y_2 = p_2$, …, $y_n = p_n$. Es wird vorausgesetzt, dasz die f_i analytische Funktionen ihrer Veränderlichen sind und also entwickelbar sind in eine Reihe nach aufsteigenden Potenzen von

$$x - x_0, y_1 - p_1, y_2 - p_2, \ldots, y_n - p_n.$$

Es mögen diese Reihen konvergieren für

$$|x - x_0| < r_x, |y_1 - p_1| < r_{y_1}, \ldots, |y_n - p_n| > r_{y_n}.$$

Es sei auszerdem

$$\bar{r}_x < r_x, \bar{r} < r_{y_1}, \ldots, \bar{r} < r_{y_n},$$

also

$$\bar{r} < \min(r_{y_1}, \ldots, r_{y_n}).$$

Der Einfachheit wegen setzen wir

$$x - x_0 = \xi, y_1 - p_1 = \eta_1, \ldots, y_n - p_n = \eta_n,$$

und die Reihen der f_i verwandeln sich in

$$f_i = \sum_{\mu, \ldots} a_{i,\mu \ldots}\, \xi^{\mu}\, \eta_1^{\mu_1} \ldots \eta_n^{\mu_n} \qquad (i = 1, \ldots, n),$$

worin $a_{i,\mu\ldots}$ Elemente aus einem vollständigen bewerteten Körper sind. Die Lösungen der Differentialgleichungen

$$\eta_i' = \sum_{\mu, \ldots} a_{i,\mu \ldots}\, \xi^{\mu}\, \eta_1^{\mu_1} \ldots \eta_n^{\mu_n} \qquad (i = 1, 2, \ldots, n)$$

(mit den Anfangsbedingungen $\eta_1 = 0, \ldots$ für $\xi = 0$) schreiben wir in der Form einer Reihe

$$\eta_i = F_i(\xi) = \sum_{k=1}^{\infty} \frac{1}{k!}\, c_{ik}\, \xi^k \qquad (i = 1, 2, \ldots, n),$$

deren Koeffizienten c_{ik} auf eindeutige Weise durch Koeffizientenvergleichen bestimmt werden. Die Konvergenz wird mit Hilfe der Majorantenmethode bewiesen. Für jedes F_i ist die Funktion

$$\frac{M}{\left(1 - \frac{\Xi}{\bar{r}_x}\right)\left(1 - \frac{\mathrm{H}_1}{\bar{r}}\right)\left(1 - \frac{\mathrm{H}_2}{\bar{r}}\right) \cdots \left(1 - \frac{\mathrm{H}^n}{\bar{r}}\right)}$$

eine Majorante (worin M eine obere Schranke ist für alle Glieder der Reihe

$$\sum_{\mu, \ldots} |a_{i,\mu,\ldots}|\, \bar{r}_x^{\mu}\, \bar{r}^{\mu_1} \ldots \bar{r}^{\mu_n} \quad \textit{für} \quad (i = 1, 2, \ldots, n).$$

Wir haben nun das System der Differentialgleichungen zu betrachten:

$$\mathrm{H}_i' = \frac{M}{\left(1 - \frac{\Xi}{\bar{r}_x}\right)\left(1 - \frac{\mathrm{H}_1}{\bar{r}}\right) \cdots \left(1 - \frac{\mathrm{H}_n}{\bar{r}}\right)} \qquad (i = 1, 2, \ldots, n)$$

mit der Anfangsbedingung

$$\Xi = 0, \ \mathrm{H} = 0.$$

Diese Lösungen, welche alle dieselbe erste Ableitung besitzen und für $\Xi = 0$ verschwinden, sind identisch, also brauchen wir nur die Differentialgleichung

$$\mathrm{H}' = \frac{M}{\left(1 - \frac{\Xi}{\bar{r}_x}\right)\left(1 - \frac{\mathrm{H}}{\bar{r}}\right)^n} \qquad (\text{mit } \Xi = 0, \ \mathrm{H} = 0)$$

zu lösen. Trennen wir die Veränderlichen, so erhalten wir

$$\mathrm{H} = \bar{r} - \bar{r}\sqrt[n+1]{1 + \frac{(n+1)M\bar{r}_x}{r}\left(\log\left(1 - \frac{\Xi}{\bar{r}_x}\right)\right)},$$

und es ist $\mathrm{H} = 0$ für $\Xi = 0$. Also ergibt sich den Konvergenzradius

$$\varrho = \bar{r}_x\left(1 - e^{-\frac{\bar{r}}{(n+1)M\bar{r}_x}}\right).$$

Für $|x - x_0| < \varrho$ besitzt also das System von Differentialgleichungen (1) eindeutig bestimmte Lösungen.

§ 4. Wenn

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n, a_1, \ldots, a_m) \qquad (i = 1, 2, \ldots, n),$$

ein simultanes System von Differentialgleichungen (mit Anfangsbedingungen), analytisch abhängig ist von seinen Parametern, so hängen auch die Lösungen analytisch von diesen Parametern und den Anfangswerten ab. Wir vereinfachen den Beweis in dem Sinne, dasz wir zeigen, dasz die Differentialgleichung

$$\frac{dy}{dx} = f(x, y, a), \tag{1}$$

worin $f(x, y, a)$ eine analytische Funktion der x, y und a darstellt, eine Lösung

$$y = F(x, a)$$

besitzt, so dasz zunächst $F(x, a)$ ebenfalls eine analytische Funktion von x und a ist, die für $x = x_0$ den Wert $y = y_0$ annimmt.

Aus dem Beweis geht sofort hervor, dasz man ihm leicht auf dem allgemeinen Fall übertragen kann. Setzen wir zuerst

$$\frac{d\eta}{d\xi} = f(\xi + x_0, \eta + y_0, a) = \varphi(\xi, \eta, a), \tag{2}$$

worin nach Voraussetzung f, also φ eine analytische Funktion der Veränderlichen ξ, η und a darstellt und entwickelbar ist in eine Reihe, welche für

$$|\xi| < R_1, |\eta| < R_2, |a| < R_3$$

konvergiert. Wir bezeichnen diese Reihe mit „Reihe (2)". Wir fragen nun, wie die Lösung

$$\eta = c_1\xi + c_2\xi^2 + \ldots \tag{3}$$

sich als Funktion von α verhält. Wir denken uns die Koeffizienten c_i berechnet mit Hilfe der Differentialgleichung. Aus dem obigen ergibt sich, dasz die c_i sich als analytische Funktionen der Koeffizienten von $\varphi(\xi, \eta, \alpha)$ ergeben. Daraus schlieszt man aber doch nicht, dasz auch (3) sich in eine analytische Funktion von x und α umordnen läszt. Es läszt sich dies aber durch eine Erweiterung der Majorantenmethode feststellen. Es sei

$$R'_1 < R_1, \; R'_2 < R_2, \; R'_3 < R_3.$$

„Reihe (2)" konvergiert für

$$|\xi| = R'_1, |\eta| = R'_2, |\alpha| = R'_3,$$

und es sei M eine obere Schranke der Bewertungen der Glieder jener Reihe für $|\xi| = R'_1$, $|\eta| = R'_2$ und $|\alpha| = R'_3$. Wir nehmen dann als Majorante die Funktion

$$g(X, Y, A) = \frac{M}{\left(1 - \frac{X}{R'_1}\right)\left(1 - \frac{Y}{R'_2}\right)\left(1 - \frac{A}{R'_3}\right)} \tag{4}$$

(X, Y und A sind nur reeller Werte fähig) mit der zugehörigen Differentialgleichung

$$\frac{dY}{dX} = \frac{1}{\left(1 - \frac{X}{R'_1}\right)\left(1 - \frac{Y}{R'_2}\right)} \cdot \frac{M}{1 - \frac{A}{R'_3}}, \tag{5}$$

die sich also von der in § 2 betrachteten Differentialgleichung nur dadurch unterscheidet, dasz

$$\frac{M}{1 - \frac{A}{R'_3}}$$

an die Stelle von M getreten ist. Ist nun $R''_3 < R'_3 < R_3$, und setzen wir

$$M' = \frac{M}{1 - \frac{R''_3}{R'_3}}$$

so konvergiert die zugehörige Reihe Y, die die für $X = 0$ verschwindende Lösung der Hilfsdifferentialgleichung (5) darstellt, für

$$\| X \| < \varrho' = R'_1 \left(1 - e^{-\frac{R'_2}{2M'R'_1}}\right), \; \| A \| < R''_3.$$

Diese Reihe hat, wenn man sie nach Potenzen von X und A ordnet, lauter positive Koeffizienten, und diese sind niemals kleiner als die Bewertungen der entsprechenden Koeffizienten der Reihe, die aus (3) durch Umordnung

nach Potenzen von x und y hervorgeht. Daraus folgt also die absolute Konvergenz der so umgeordneten Reihe für

$$|x| < \varrho', \ |\alpha| < R_3'' \tag{6}$$

d.h. die für $\xi = 0$ verschwindende Lösung für η unserer Differentialgleichung (2) ist unter den Bedingungen (6) eine analytische Funktion von ξ und α.

Aus dem obigen geht nun hervor, dasz die Differentialgleichung

$$\frac{dy}{dx} = f(x, y), \tag{7}$$

worin $f(x, y)$ eine analytische Funktion der Veränderlichen x und y mit Koeffizienten aus einem vollständigen bewerteten Körper darstellt, eine Lösung

$$y = F(x, x_0, y_0)$$

besitzt, die eine analytische Funktion der drei Veränderlichen x, x_0 und y_0 darstellt und die für $x = x_0 = 0$ die Wert $y_0 = 0$ annimmt.

Setzt man $x = \xi + x_0$, $y = \eta + y_0$, so genügt η der Differentialgleichung

$$\frac{d\eta}{d\xi} = f(\xi + x_0, \eta + y_0),$$

und die Lösung η, die für $\xi = 0$ verschwindet,

$$\eta = F(\xi + x_0, x_0, y_0)$$

ist nach dem oben bewiesenen Satze eine analytische Funktion der Veränderlichen ξ, x_0 und y_0 (in der Umgebung von $\xi = 0$, $x_0 = 0$, $y_0 = 0$). Also ist η nach positiven ganzen Potenzen von $x - x_0$, x_0 und y_0 entwickelbar und es ist $y = F(x, x_0, y_0)$ „in der Umgebung" von $x = x_0 = 0$ und $y_0 = 0$ analytisch. Hieraus ergibt sich also, dasz die Lösung

$$y = F(x, x_0, y_0)$$

eine analytische Funktion der drei Veränderlichen x, x_0 und y_0 ist.

§ 5. Betrachten wir zum Schlusz kurz die partiellen Differentialgleichungen. Mit Hilfe der Majorantenmethode sind wir imstande, die Existenz der Lösungen eines Systems von partiellen Differentialgleichungen zu bestimmen. Wir betrachten zunächst eine Gleichung der Form

$$\frac{\partial z}{\partial x_1} = f\left(x_1, x_2, \ldots . x_n, z, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n}\right), \tag{1}$$

worin f die Ableitung $\frac{\partial z}{\partial x_1}$ nicht enthält und eine analytische Funktion von

$$x_1, x_2, \ldots, x_n, z, p_2, \ldots, p_n$$

darstellt, und versuchen eine ebenfalls analytische Funktion der Veränderlichen $x_1, x_2, \ldots, x_n$ zu bestimmen, welche in eine Reihe nach aufsteigenden Potenzen von $x_1, \ldots, x_n$ zu entwickeln ist und für $x_1 = 0$ den analytischen Ausdruck $\varphi(x_2, \ldots, x_n)$ darstellt und die Differentialgleichung formal befriedigt.

Es ist klar, dasz, wenn wir für z eine Reihe

$$z = \sum_{\mu_i} a_{\mu_1 \ldots \mu_n} x_1^{\mu_1} \ldots x_n^{\mu_n} \tag{2}$$

ansetzen, sich durch Differenzieren und Koeffizientenvergleichen herausstellt, dasz die Koeffizienten $a_{\mu_1 \ldots \mu_n}$ nur mittels die Operationen der Addition und Multiplikation zusammengesetzt sind aus denen von f. Es handelt sich noch um die Frage ihrer Konvergenz.

Dazu benutzen wir auch hier die Majorantenmethode: wenn wir für f eine Majorante F und für φ eine Majorante Φ wählen (mit reellen Koeffizienten u.s.w.) und aufs neue eine Reihe bestimmen, die der Differentialgleichung

$$\frac{\partial Z}{\partial X_1} = F\left(X_1, X_2, \ldots, X_n, Z, \frac{\partial Z}{\partial X_2}, \ldots, \frac{\partial Z}{\partial X_n}\right)$$

(mit Anfangsbedingungen) genügt, so folgt aus der Konvergenz der Majorante die Konvergenz der Reihe (2).

Wir setzen voraus, dasz f kein konstantes Glied enthält (sonst nehmen wir eine Transformation $\bar{x}_1 = a x_1 + b$ vor).

Ersetzen wir nun f durch eine Majorante, so bleibt nur zu beweisen, dasz die Gleichung

$$\frac{\partial Z}{\partial X_1} = \frac{M}{\left(1 - \frac{X_1 + X_2 + \ldots X_n + Z}{r}\right)\left(1 - \frac{\frac{\partial Z}{\partial X_2} + \ldots + \frac{\partial Z}{\partial X_n}}{\varrho}\right)} - M \tag{3}$$

worin M, r und ϱ bestimmte positive Gröszen sind, in eine Reihe zu entwickeln ist; ersetzen wir auszerdem X_1 durch $\frac{X_1}{\alpha}$, worin $0 < \alpha < 1$, so werden die Koeffizienten nur vergröszert.

Wir versuchen für (2) eine Lösung der einzigen Veränderlichen

$$X = \frac{X_1}{\alpha} + X_2 + \ldots + X_n$$

zu bestimmen.

$$\left(\frac{1}{\alpha} - \frac{n-1}{\varrho} M\right)\frac{dZ}{dX} = \frac{n-1}{\varrho}\left(\frac{dZ}{dX}\right)^2 + \frac{M}{1 - \frac{X+Z}{r}} - M. \tag{4}$$

Sei α so bestimmt, dasz der Koeffizient von $\frac{dZ}{dX}$ positiv ist. Für $X = Z = 0$ hat diese Gleichung zwei verschiedene Wurzeln, wovon die eine Null ist.

Diese Gleichung besitzt also ein Integral, welches in einer Reihe darzustellen ist, und ebenso wie die erste Ableitung für $X = 0$ verschwindet. Man überzeugt sich einfach davon, dasz die Koeffizienten dieser Reihe positive Zahlen sind: es ist ja die Gleichung (4) in die Form zu schreiben:

$$\frac{dZ}{dX} = A\left(\frac{dZ}{dX}\right)^2 + g(X, Z)$$

und es ist A positiv und $g(X, Z)$ eine Reihe, von der alle Koeffizienten positiv sind.

Damit ist eine Reihe für Z bestimmt, welche für bestimmtes X konvergiert, aber damit ist zugleich die Existenz der Lösung von (1) bewiesen worden. Diese Beweisführung bleibt dieselbe für ein simultanes System von Gleichungen der ersten Ordnung

$$\frac{\partial z_1}{\partial x_1} = f_1, \ldots, \frac{\partial z_p}{\partial x_1} = f_p \quad \text{(mit Anfangsbedingungen).}$$

Mathematics. — *Elemente der intuitionistischen Funktionentheorie.* (Dritte Mitteilung)[1]. *Der Satz vom Integral der logarithmischen Ableitung.* II. By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of March 29, 1941.)

§ 1. **Hauptsatz.** *Es sei $f(z)$ regulär für $|z| < R$, G ein in $|z| < R$ liegendes einfach zusammenhängendes Gebiet auf dessen Rande $f(z) \neq 0$ ist und*

$$I_G = \frac{1}{2\pi\sqrt{-1}} \int_G \frac{f'(z)\,dz}{f(z)} = m > 0.$$

Alsdann kann man in G m Punkte $w_1, w_2, \ldots w_m$ so bestimmen, dass:

a. $f(w_i) = 0$ *ist,*

b. $\lim\limits_{z \to w_i} \dfrac{f(z)}{\prod\limits_{k=1}^{m}(z - w_k)}$ *existiert und*

c. *die Funktion* $\varphi(z) = \dfrac{f(z)}{\prod\limits_{k=1}^{m}(z - w_k)}$ *in G regulär und $\neq 0$ ist.*[2])

Bemerkung. Die Behauptungen (*a*), (*b*) und (*c*) kann man zusammenfassen in der Behauptung, dass $f(z)$ in G m und nur m Nullstellen hat. Hiermit ist natürlich keineswegs gesagt, dass $w_1, w_2, \ldots w_m$ von einander verschieden sind, noch dass eine Entscheidung über die Frage ob oder welche Punkte w_i zusammenfallen möglich ist. Wir werden im folgenden die Funktion $P(z) = \prod\limits_{k=1}^{m}(z - w_k)$ das *Nullstellenpolynom* von $f(z)$ in G nennen.

Weiter bemerken wir, dass der vorliegende Satz eine intuitionistische Uebertragung des klassischen Satzes über das logarithmische Residuum ist. Jener Satz lautet bekanntlich, dass das Integral $\dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_G \frac{f'(z)\,dz}{f(z)}$ für eine in $|z| < R$ bis auf endlich viele Pole reguläre Funktion $f(z)$ der um die Anzahl der Pole verminderte Anzahl der Nullstellen von $f(z)$ innerhalb G gleich ist. Für den Fall, dass man genau über die

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[1]) Vgl. Proc. Ned. Akad. v. Wetensch., Amsterdam, Bd. 44, S. 173 und 276 (1941).

[2]) Es sind (*a*) und (*b*) natürlich unmittelbare Folgerungen von (*c*).

Lage und die Ordnung dieser Pole und Nullstellen orientiert ist, bleibt der klassische Beweis gültig. Handelt es sich aber darum, die Existenz von Nullstellen darzutun, so ist ein neuer, intuitionistischer Beweis erforderlich. In dem Hauptsatz ist vorausgesetzt, das $f(z)$ keine Pole in $|z|<R$ hat. Der Fall, dass von $f(z)$ die Pole $z_1, z_2, \ldots, z_n$ bzw. von der Ordnung $\alpha_1, \alpha_2, \ldots, \alpha_n$ bekannt sind, lässt sich sofort auf den Hauptsatz zurückführen, indem man diesen Satz auf $F(z)=f(z)\prod\limits_{1}^{n}(z-z_i)^{\alpha_i}$ anwendet. Ist man aber *nicht* über die Lage und die Ordnung der Pole orientiert, so lässt sich im allgemeinen weder die Anzahl der Nullstellen, noch die Anzahl der Pole bestimmen. Man erkennt dies an dem Beispiel $f(z)=\frac{z^2}{z-\alpha}$, falls $|\alpha|<1$, jedoch weder $\alpha=0$ noch $\alpha\neq 0$ festgestellt ist. Es ist dann $\frac{1}{2\pi\sqrt{-1}}\int\limits_{|z|=1}\frac{f'(z)\,dz}{f(z)}=1$, aber es lässt sich weder die Anzahl der Nullstellen, noch die der Pole bestimmen.

Beweis des Hauptsatzes. Sei $\varepsilon_1, \varepsilon_2, \ldots$ eine monotone Nullfolge positiver Zahlen. Wir zerlegen G nach dem Zerlegungssatz (§ 3 der zweiten Mitteilung) in solche Teilgebiete g_{i_1}, dass $f(z)$ auf dem Rande eines Teilgebietes $\neq 0$ ist und die Entfernung zweier Punkte eines Teilgebietes $<\varepsilon_1$ ist. Setzen wir

$$I_{i_1}=\frac{1}{2\pi\sqrt{-1}}\int\limits_{g_{i_1}}\frac{f'(z)\,dz}{f(z)},$$

so ist offenbar $\sum\limits_{(i_1)} I_{i_1}=m$. Diejenigen Teilgebiete g_{i_1}, für die $I_{i_1}>0$ ist, zerlegen wir von neuem in solche Teilgebiete $g_{i_1 i_2}$, dass $f(z)$ auf dem Rande eines Teilgebietes $\neq 0$ ist und die Entfernung zweier Punkte eines Teilgebietes $<\varepsilon_2$ ist. Setzen wir

$$I_{i_1 i_2}=\frac{1}{2\pi\sqrt{-1}}\int\limits_{g_{i_1 i_2}}\frac{f'(z)\,dz}{f(z)},$$

so ist offenbar $\sum\limits_{i_1 i_2} I_{i_1 i_2}=m$. Diejenigen Teilgebiete $g_{i_1 i_2}$, für die $I_{i_1 i_2}>0$ ist, zerlegen wir von neuem usw. Auf der n-ten Stufe dieses Prozesses setzen wir

$$I_{i_1 i_2 \ldots i_n}=\frac{1}{2\pi\sqrt{-1}}\int\limits_{g_{i_1 i_2 \ldots i_n}}\frac{f'(z)\,dz}{f(z)}. \quad \ldots\ldots \quad (1)$$

Es ist dann $\sum\limits_{i_1 i_2 \ldots i_n} I_{i_1 i_2 \ldots i_n}=m$. Wir zerlegen diejenigen Teilgebiete $g_{i_1 \ldots i_n}$, für die $I_{i_1 \ldots i_n}>0$ ist, in solche Teilgebiete $g_{i_1 \ldots i_n i_{n+1}}$, dass $f(z)$ auf dem Rande eines Teilgebietes $\neq 0$ und die Entfernung zweier Punkte eines

Teilgebietes $g_{i_1\ldots i_n i_{n+1}}$ kleiner als ε_{n+1} ist, usw. Weiter wählen wir auf jeder Stufe m Punkte und zwar auf der n-ten Stufe im Innern eines Gebietes $g_{i_1\ldots i_n}$, für welches $I_{i_1\ldots i_n}>0$ ist, gerade $I_{i_1\ldots i_n}$ beliebige Punkte $z_{i_1\ldots i_n}$. Wir bilden nun aus diesen Punkten m konvergente Punktfolgen. Als ersten Punkt einer Folge wählen wir einen Punkt z_{i_1}. Als zweiten Punkt wählen wir einen Punkt $z_{i_1 i_2}$ mit demselben i_1 als der ihm vorangehende Punkt. Im allgemeinen wählen wir als n-ten Punkt einen Punkt $z_{i_1 i_2\ldots i_n}$ mit denselben $i_1\, i_2\ldots i_{n-1}$ als der ihm vorangehende Punkt. Da offenbar

$$|z_{i_1\ldots i_n}-z_{i_1\ldots i_{n+p}}|<\varepsilon_n$$

ist, konvergiert jede dieser m Punktfolgen zu einem Limespunkt. Die so bestimmten m Limespunkte $w_1, w_2, \ldots, w_m$ sind nun die gesuchten Punkte. Zum Beweise setzen wir $P(z)=\prod\limits_{i=1}^{m}(z-w_i)$. Es ist dann

$$\frac{1}{2\pi\sqrt{-1}}\int\limits_{g_{i_1\ldots i_n}}\frac{P'(z)\,dz}{P(z)}=I_{i_1\ldots i_n}, \quad \ldots\ldots \quad (2)$$

da ja $I_{i_1\ldots i_n}$ Punkte w_i innerhalb $g_{i_1\ldots i_n}$ liegen. Setzen wir weiter $\varphi(z)=\dfrac{f(z)}{P(z)}$, so ist $\varphi(z)$ regulär in jedem Teilgebiet von G, das keinen Punkt w_i enthält. In einem von jedem Punkte w_i entfernten Punkte z ist $f(z)\neq 0$, wie man leicht mit Hilfe von Satz VI der zweiten Mitteilung folgert. Wir haben wegen (1) und (2):

$$\int\limits_{g_{i_1\ldots i_n}}\frac{\varphi'(z)\,dz}{\varphi(z)}=0,$$

also auch

$$\int\limits_{W}\frac{\varphi'(z)\,dz}{\varphi(z)}=0, \quad \ldots\ldots\ldots \quad (3)$$

wenn der in G liegende, geschlossene Weg W von jedem Punkt w_i eine Entfernung $\neq 0$ hat. Nach § 3 Satz III B können wir für jeden Punkt w_i zwei positive Zahlen $r_1=r_1(w_i)$ und $r_2=r_2(w_i)$ so bestimmen, dass $f(z)$ in dem Kreisring $r_1\leqq|z-w_i|\leqq r_2$ (K_i) von Null verschieden ist. Sei m_i die Anzahl der Limespunkte w_i, welche in dem Kreis $|z-w_i|\leqq r_1$ (C_i) liegen. In K_i gilt nach LAURENT:

$$\varphi(z)=\sum_0^{\infty}a_n(z-w_i)^n+\sum_1^{\infty}b_n(z-w_i)^{-n}.$$

Wir werden nun zeigen, dass man den Kreisring K_i so wählen kann,

dass die Koeffizienten b_n *alle* Null sind [3]). Hiermit ist unsere Behauptung dargetan, denn wegen $\varphi(z) = \sum\limits_0^\infty a_n (z - w_i)^n$ existiert dann $\lim\limits_{z \to w_i} \varphi(z)$ und ist $\varphi(z)$ in $|z - w_i| < r_1(w_i)$, also überall in G regulär. Auch ist dann nach Satz VI der zweiten Mitteilung $\varphi(z) \neq 0$.

Der nun noch fehlende Beweis, dass die Koeffizienten b_n für einen passend gewählten Kreisring verschwinden, führen wir in zwei Etappen. Wir zeigen nacheinander:

(A). *Ist* $b_{m_i+p} \neq 0\ (p > 0)$, *so lässt sich in* C_i *ein von* w_i *entfernter Limespunkt* w_j *bestimmen.*

(B). *Ist* $b_{m_i+p} = 0$ *für jedes* $p > 0$ *und ist* $b_k \neq 0$ *für ein gewisses* $k \leqq m_i$, *so lässt sich in* C_i *ein von* w_i *entfernter Limespunkt* w_j *bestimmen.*

Sobald aber in C_i ein von w_i entfernter Limespunkt w_j bestimmt ist, können wir einen neuen Kreisring um w_i wählen, dessen innerer Kreis den Punkt w_j *nicht* enthält. Aus (A) folgt also nach höchstens m_i-facher Wiederholung, dass für einen passend gewählten Kreisring um w_i *alle* b_{m_i+p} mit $p > 0$ Null sind. Darauf folgt aber aus (B) in derselben Weise, dass *alle* Koeffizienten b_k Null sind.

Beweis von (A). Sei $|b_{m_i+k}| > \alpha$ (4), $|f(z)|$ in G kleiner als μ, $M = 2^m r_1^{m_i - m}$ und β eine positive Zahl kleiner als jede der Zahlen $\frac{1}{2}$, $\frac{1}{4} r_1$ und $\frac{\alpha}{2\mu M}$. Für die in C_i liegenden Limespunkte w_j gilt nun:

entweder $|w_i - w_j| > \frac{1}{2}\beta$ für wenigstens ein bestimmtes $j \neq i$, *oder* $|w_i - w_j| < \beta$ für *alle* $j \neq i$.

Im ersten Fall gibt es in der Tat in C_i einen von w_i entfernten Limespunkt w_j; im zweiten Fall aber wäre $\varphi(z)$ auch noch regulär für $\beta \leqq |z - w_i| \leqq r_1$ und hätten wir für $|z - w_i| = 2\beta$:

$$\left| \frac{(z - w_i)^{m_i}}{P(z)} \right| < \frac{(2\beta)^{m_i}}{\beta^{m_i} \left(\frac{r_1}{2} \right)^{m - m_i}} = M,$$

mithin:

$$|b_{m_i+k}| = \left| \frac{1}{2\pi\sqrt{-1}} \int\limits_{|z - w_i| = 2\beta} \frac{f(z)}{P(z)} (z - w_i)^{m_i + k - 1}\, dz \right| < \mu M (2\beta)^k < \alpha,$$

entgegen der Annahme (4).

Beweis von (B). Da $b_k \neq 0$ ist, lässt sich offenbar ein positives $\delta < r_1$ so bestimmen, dass

$$|b_k|\, \delta^{-k} > 4 \sum_0^\infty |a_p|\, \delta^p + 4 \sum_1^{k-1} |b_p|\, \delta^{-p} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[3]) Hinterher erkennen wir, dass die Koeffizienten b_n für *jede* Wahl von K_i verschwinden.

ist. Man hat nun *entweder*:

$$\sum_{k+1}^{m_i} |b_p|\,\delta^{-p} > \tfrac{1}{8}\,|b_k|\,\delta^{-k}$$

und dann gibt es ein $b_l \neq 0$ $(k < l \leqq m_i)$, mit dem wir die Untersuchung von neuem anfangen, *oder*:

$$\sum_{k+1}^{m_i} |b_p|\,\delta^{-p} < \tfrac{1}{4}\,|b_k|\,\delta^{-k} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Weiter haben wir für die in C_i liegenden Limespunkte w_j *entweder* $|w_i - w_j| > \frac{1}{2}\delta$ für wenigstens ein bestimmtes $j \neq i$, *oder* $|w_i - w_j| < \delta$ für jedes $j \neq i$.

Im ersten Fall gibt es in der Tat in C_i einen von w_i entfernten Limespunkt w_j; im zweiten Fall aber wäre $\varphi(z)$ auch noch regulär in dem Kreisring $\delta \leqq |z - w_i| \leqq r_1$. Für jedes z mit $|z - w_i| = \delta$ wäre nun wegen (5) und (6):

$$|\varphi(z) - b_k(z - w_i)^{-k}| < \tfrac{1}{2}\,|b_k(z - w_i)^{-k}|.$$

Folglich wäre

$$\frac{1}{2\pi\sqrt{-1}} \int\limits_{|z-w_i|=\delta} \frac{\varphi'(z)\,dz}{\varphi(z)} = -k,$$

entgegen der Ungleichung (3).

§ 2. Zu den Sätzen der zweiten Mitteilung und zu dem Hauptsatz gibt es Nebensätze, wobei der Kreis $|z| = R$ bzw. $|z - z_0| = R$ durch den Ring $R_1 \leqq |z| \leqq R_2$ bzw. $|R_1| \leqq |z - z_0| \leqq R_2$ ersetzt wird und entsprechende Abänderungen in den Behauptungen vorzunehmen sind. Diese Nebensätze können in derselben Weise wie die verwandten Sätze oder durch Anwendung dieser Sätze gefolgert werden.

Wir beschränken uns auf die Erwähnung einiger dieser Nebensätze.

Nebensatz zu Satz III der zweiten Mitteilung. *Es sei $f(z)$ eindeutig-regulär und variabel in dem Ring $R_1 \leqq |z| \leqq R_2$ und es seien die positiven Zahlen $d < \dfrac{R_2 - R_1}{2}$ und ε beliebig vorgelegt. Alsdann lässt sich eine positive Zahl $k < \varepsilon$ mit folgender Eigenschaft bestimmen:*

(*A*). *Zu jedem z_0 in dem Ring $R_1 + d \leqq |z| \leqq R_2 - d$ lässt sich ein z_1 in dem Ring $R_1 + \frac{1}{2}d \leqq |z| \leqq R_2 - \frac{1}{2}d$ so bestimmen, dass $|f(z_1)| > |f(z_0)| + k$ ist.*

(*B*). *Jedem z_0 in dem Ring $R_1 + d \leqq |z| \leqq R_2 - d$ kann man zwei zwischen k und ε liegende positive Zahlen r_1 und r_2 so zuordnen, dass*

$f(z)$ in dem Ring $r_1 \leqq |z - z_0| \leqq r_2$ dem absoluten Betrage nach grösser als k ist.

Nebensatz zu Satz VI der zweiten Mitteilung. *Es sei $f(z)$ eindeutig-regulär in dem Ring $R_1 \leqq |z| \leqq R_2$. Es seien dort zwei Polygone L und l (l innerhalb L) vorgelegt, auf deren Rändern $|f(z)| > k > 0$ (1) ist und es sei*

$$\int_L \frac{f'(z)\,dz}{f(z)} - \int_l \frac{f'(z)\,dz}{f(z)} = 0.$$

Alsdann gilt (1) in dem ganzen von L und l gebildeten Polygonring.

Nebensatz zu Satz VII der zweiten Mitteilung. *Es sei $f(z)$ eindeutig-regulär für $R_1 \leqq |z| \leqq R_2$ und es sei $|f(z)| \leqq M$ (1) sowohl für $|z| = R_1$ als für $|z| = R_2$. Alsdann gilt (1) auch für $R_1 < |z| < R_2$.*

Nebensatz zum Hauptsatz. *Es sei $f(z)$ eindeutig-regulär in dem Ring $R_1 \leqq |z| \leqq R_2$. Es seien dort zwei Polygone L und l (l innerhalb L) vorgelegt, auf deren Rändern $f(z) \neq 0$ ist und es sei:*

$$\frac{1}{2\pi\sqrt{-1}} \int_L \frac{f'(z)\,dz}{f(z)} - \frac{1}{2\pi\sqrt{-1}} \int_l \frac{f'(z)\,dz}{f(z)} = m > 0.$$

Alsdann hat $f(z)$ m Nullstellen in dem von L und l gebildeten Polygonring.

Mathematics. — *Diophantische Approximationen homogener Linearformen in imaginären quadratischen Zahlkörpern.* Von J. F. KOKSMA und B. MEULENBELD. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 29, 1941.)

§ 1. *Einleitung.*

Sind $\theta_1, \theta_2, \ldots, \theta_n$ irgend n $(n \geqq 1)$ reelle Zahlen, so kann man, wie wir in einer vorigen Arbeit mit Hilfe der BLICHFELDTschen zahlengeometrischen Methode gezeigt haben, unendlich viele verschiedene Systeme ganzer rationaler Zahlen $x_1, x_2, \ldots, x_n, y$ mit

$$X = max. (|x_1|, |x_2|, \ldots, |x_n|) \geqq 1$$

finden, für die die Linearform

$$L = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n - y$$

der Ungleichung

$$|L| \leqq \frac{1}{\varrho_n X^n}$$

mit

$$\varrho_n = \left(1 + \frac{1}{n}\right)^n \left\{1 + \frac{2^{n+1} n^{n+1}}{(n+1)^n} \sum_{\mu=n+2}^{\infty} \frac{1}{\mu} \left(\frac{n-1}{2n}\right)^\mu \right\}$$

genügt[1]). In der vorliegenden Arbeit wollen wir diesen Satz ins Komplexe übertragen.

Sei m eine quadratfreie natürliche Zahl und

$$\left.\begin{array}{l} \chi = 0, \lambda = 1 \text{ für } m \not\equiv 3 \,(mod\, 4) \\ \chi = 1, \lambda = 2 \text{ für } m \equiv 3 \,(mod\, 4) \end{array}\right\} \quad \ldots \ldots \quad (1)$$

gesetzt. Dann bildet bekanntlich $\left(1, \frac{\chi + i\sqrt{m}}{\lambda}\right)$ eine Basis des Körpers

[1]) J. F. KOKSMA und B. MEULENBELD, Ueber die Approximation einer homogenen Linearform an die Null. Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, S. 62—74 (1941). Die obendefinierte Zahl ϱ_n ist gleich dem γ_n der Formel (9) jener Arbeit: man führe daselbst $w = \frac{v}{1+v}$ als neue Integrationsvariable ein, entwickle den Integrand in einer Potenzreihe nach w und integriere gliedweise.

$K(i\sqrt{m})$, so dass das System der ganzen Zahlen aus $K(i\sqrt{m})$ mit dem System der Zahlen

$$X = x + \frac{\chi + i\sqrt{m}}{\lambda}\, y \qquad (x, y \text{ ganz rational})$$

identisch ist.

Wir beweisen nun in § 2 den folgenden

Satz 1. *Für ganze rationale* $n \geqq 1, m \geqq 1$ (*m quadratfrei*) *werde* λ *durch* (1) *und* $\varrho_{n,m}$ *durch*

$$= \frac{2^{n+1} n^n}{(n+1)^n} \sqrt{\frac{(2n+1)\, m^{\frac{n+1}{2}}}{(n+1)\pi^{n+1}\lambda^{n+1}}} \sqrt{\frac{1}{1 + \frac{(n-1)^{2n+2}}{(n+1)^{2n+1} n} + \frac{2^{2n+3} n^{2n+1}(2n+1)}{(n+1)^{2n+1}} \sum\limits_{\mu=2n+3}^{\infty} \frac{1}{\mu}\left(\frac{n-1}{2n}\right)^{\mu}}} \quad (2)$$

definiert. Ferner seien $\theta_1, \theta_2, \ldots, \theta_n$ *beliebige komplexe Zahlen und* t *eine reelle Zahl* > 2. *Dann kann man immer ganze Zahlen* $P_1, P_2, \ldots P_n, Q$ *aus* $K(i\sqrt{m})$ mit

$$1 \leqq P = \max.(|P_1|, |P_2|, \ldots, |P_n|) \leqq 2(\varrho_{n,m}\, t)^{\frac{1}{n}} \quad . \quad . \quad (3)$$

finden, so dass die Linearform

$$L = \theta_1 P_1 + \theta_2 P_2 + \ldots + \theta_n P_n - Q$$

den Ungleichungen

$$|L| \leqq \frac{\varrho_{n,m}}{P^n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

und

$$|L| \leqq \frac{2}{t} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

genügt.

Bemerkungen. 1. Es genügt den Satz zu zeigen mit

$$1 \leqq P < (\varrho_{n,m}\, t)^{\frac{1}{n}}(2+\delta), \quad |L| < \frac{\varrho_{n,m}(1+\delta)^n}{P^n}, \quad |L| < \frac{2+\delta}{t}$$

für jedes beliebige positive δ, statt mit (3), (4) und (5), denn weil diesen Ungleichungen höchstens endlich viele Systeme ganzer $(P_1, P_2, \ldots, P_n, Q)$ aus $K(i\sqrt{m})$ genügen können, folgt aus Stetigkeitsgründen sofort der Satz.

2. Aus Satz 1 folgt, dass es zu jedem $t > 2$ ein System ganzer $(P_1, P_2, \ldots, P_n, Q)$ aus $K(i\sqrt{m})$ mit (3), (4), (5) gibt. Ist nun L eine *eigentliche* Form, d.h. ist L für jedes solche System $\neq 0$, so kann, wie man sofort aus (5) ersieht, das betreffende System bei unbeschränkt wachsendem t nicht immer dasselbe bleiben, sondern es muss vielmehr die Zahl P mit t unbeschränkt wachsen. Es gibt also unendlich viele Systeme ganzer $(P_1, P_2, \ldots, P_n, Q)$ mit (4). Im Fall, dass L *uneigentlich* ist, ist die letzte

Aussage trivial, denn mit $(P_1, P_2, \ldots, P_n, Q)$ ist für ganzes Z aus $K(i\sqrt{m})$ offenbar auch $(ZP_1, ZP_2, \ldots, ZP_n, ZQ)$ eine Lösung von (4).

3. Das mit dem Gegenstand dieser Note eng zusammenhängende Problem der simultanen Approximation komplexer Zahlen durch Brüche (gleichen Nenners) aus $K(i\sqrt{m})$ haben wir in einer andren Abhandlung behandelt [2]).

4. Auch beim Beweis des Satzes 1 benutzen wir die BLICHFELDTsche Methode, indem wir uns stützen auf den folgenden BLICHFELDTschen [3])

Satz 2. *Der Raum R_M der Punkte $(u_1, u_2, \ldots, u_M)$ $(M \geqq 2)$ werde durch die „Ebenen"*

$$u_\mu = a_\mu + b_\mu t \; (\mu = 1, 2, \ldots, M;\; t = 0, \pm 1, \pm 2, \ldots;\; a_\mu, b_\mu \text{ reell, fest})$$

in Fundamentalparallelepipede R eingeteilt. In jedem R seien k $(k \geqq 1)$ beliebige Punkte fest gegeben. Diese Punkte heissen hier die Gitterpunkte des Raumes R_M. Der Inhalt von R sei W. Ist nun S eine beliebige beschränkte offene stetig zusammenhängende Punktmenge im R_M mit äusserem Volumen V und ist $\varepsilon > 0$, so kann man durch eine passende Translation die Menge S immer in eine solche Lage bringen, dass die Anzahl der Gitterpunkte, welche innerhalb von S oder innerhalb einer ε-Umgebung eines Randpunktes von S liegen, grösser als $\frac{Vk}{W}$ ist.

§ 2. *Beweis des Satzes 1.* Wir setzen $\theta_\nu = \alpha_\nu + i\beta_\nu$ $(\nu = 1, 2, \ldots, n)$ und bilden den Raum R_{2n+2} der Punkte

$$(x_1, x_2, \ldots, x_n, x_{n+1}, y_1, y_2, \ldots, y_n, y_{n+1})$$

auf den Raum R'_{2n+2} der Punkte $(u_1, u_2, \ldots, u_n, u_{n+1}, v_1, v_2, \ldots, v_n, v_{n+1})$ durch die Transformation

$$\left.\begin{array}{l} \left.\begin{array}{l} u_\nu = x_\nu + \dfrac{\chi}{\lambda} y_\nu \\[2ex] v_\nu = \dfrac{\sqrt{m}}{\lambda} y_\nu \end{array}\right\} (\nu = 1, 2, \ldots, n) \\[4ex] u_{n+1} = \displaystyle\sum_{\nu=1}^{n} \left\{ \alpha_\nu x_\nu + \left(\alpha_\nu \frac{\chi}{\lambda} - \beta_\nu \frac{\sqrt{m}}{\lambda}\right) y_\nu \right\} - x_{n+1} - \frac{\chi}{\lambda} y_{n+1} \\[4ex] v_{n+1} = \displaystyle\sum_{\nu=1}^{n} \left\{ \beta_\nu x_\nu + \left(\beta_\nu \frac{\chi}{\lambda} + \alpha_\nu \frac{\sqrt{m}}{\lambda}\right) y_\nu \right\} - \frac{\sqrt{m}}{\lambda} y_{n+1} \end{array}\right\} . \quad (6)$$

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[2]) J. F. KOKSMA und B. MEULENBELD, Simultane Approximationen in imaginären quadratischen Zahlkörpern. Proc. Ned. Akad. v. Wetensch., Amsterdam, 44, S. 310—323 (1941).

[3]) H. F. BLICHFELDT, A new principle in the geometry of numbers, with some applications. Trans. Amer. Math. Soc. 15, S. 227—235 (1914).

ab. Setzt man in (6) für $(x_1, x_2, \ldots, x_n, x_{n+1}, y_1, y_2, \ldots, y_n, y_{n+1})$ sämtliche Gitterpunkte des R_{2n+2} ein, so erhält man im R'_{2n+2} ein System von Punkten $(u_1, u_2, \ldots, u_n, u_{n+1}, v_1, v_2, \ldots, v_n, v_{n+1})$, welche wir als „Gitterpunkte" des R'_{2n+2} im Sinne des Satzes 2 auffassen wollen. Weil diese Punkte ein abzählbares System bilden, lässt sich im R'_{2n+2} ein Punkt $(a_1, a_2, \ldots, a_n, a_{n+1}, b_1, b_2, \ldots, b_n, b_{n+1})$ finden, der i.B. auf alle Koordinaten von sämtlichen „Gitterpunkten" des R'_{2n+2} verschieden ist. Wir betrachten im R'_{2n+2} jetzt die Fundamentalparallelepipede R, welche von den „Ebenen"

$$\left.\begin{aligned} u_\nu &= a_\nu + g_\nu \\ v_\nu &= b_\nu + \frac{\sqrt{m}}{\lambda} h_\nu \end{aligned}\right\} (\nu = 1, 2, \ldots, n), \qquad \begin{aligned} u_{n+1} &= a_{n+1} + k_1 g_{n+1} \\ v_{n+1} &= b_{n+1} + \frac{\sqrt{m}}{\lambda} k_2 h_{n+1} \end{aligned}$$

begrenzt werden; hierin sind k_1 und k_2 vorgegebene natürliche Zahlen, während $g_1, g_2, \ldots, g_n, g_{n+1}, h_1, h_2, \ldots, h_n, h_{n+1}$ unabhängig voneinander alle ganzen rationalen Zahlen durchlaufen. Jedes der R hat offenbar das Volumen

$$W = \left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} k_1 k_2, \quad \ldots \ldots \ldots \quad (7)$$

und enthält genau

$$k = k_1 k_2 \quad \ldots \ldots \ldots \ldots \quad (8)$$

„Gitterpunkte" der R'_{2n+2}, denn bei festen g_ν und h_ν genügen genau $k_1 k_2$ Systeme von ganzen rationalen Zahlen

$$x_1, x_2, \ldots, x_n, x_{n+1}, y_1, y_2, \ldots, y_n, y_{n+1}$$

den Ungleichungen

$$\left.\begin{aligned} a_\nu + g_\nu < x_\nu + \frac{\chi}{\lambda} y_\nu < a_\nu + g_\nu + 1 \\ b_\nu + \frac{\sqrt{m}}{\lambda} h_\nu < \frac{\sqrt{m}}{\lambda} y_\nu < b_\nu + \frac{\sqrt{m}}{\lambda}(h_\nu + 1) \end{aligned}\right\} (\nu = 1, 2, \ldots, n)$$

$$a_{n+1} + k_1 g_{n+1} < \sum_{\nu=1}^{n} \left\{ \alpha_\nu x_\nu + \left(\alpha_\nu \frac{\chi}{\lambda} - \beta_\nu \frac{\sqrt{m}}{\lambda}\right) y_\nu \right\} + \\ - x_{n+1} - \frac{\chi}{\lambda} y_{n+1} < a_{n+1} + k_1 (g_{n+1} + 1)$$

$$b_{n+1} + \frac{\sqrt{m}}{\lambda} k_2 h_{n+1} < \sum_{\nu=1}^{n} \left\{ \beta_\nu x_\nu + \left(\beta_\nu \frac{\chi}{\lambda} + \alpha_\nu \frac{\sqrt{m}}{\lambda}\right) y_\nu \right\} + \\ - \frac{\sqrt{m}}{\lambda} y_{n+1} < b_{n+1} + \frac{\sqrt{m}}{\lambda} k_2 (h_{n+1} + 1).$$

Sei jetzt im Raum R'_{2n+2} der Körper S' durch die Ungleichungen

$$|u_{n+1} + i v_{n+1}| < \frac{1}{t}$$

$$\frac{|u_\nu + i v_\nu|}{a} + \frac{2^{n+1} n^n |u_{n+1} + i v_{n+1}| t}{(n+1)^{n+1}} < 1 \text{ für } |u_{n+1} + i v_{n+1}| t \leqq \left(\frac{n+1}{2n}\right)^n$$

$$|u_{n+1} + i v_{n+1}| t \left\{\frac{|u_\nu + i v_\nu|}{a} + 1\right\}^n < 1 \text{ für } \left(\frac{n+1}{2n}\right)^n < |u_{n+1} + i v_{n+1}| t < 1,$$

$$(\nu = 1, 2, \ldots, n)$$

wo zur Abkürzung

$$a = (t \varrho_{n,m})^{\frac{1}{n}} \quad \ldots \ldots \ldots \quad (9)$$

gesetzt wurde, definiert. Das Volumen von S' sei V.
Durch die Transformation

$$\left.\begin{aligned} u_\nu &= at\, u_\nu^* \\ v_\nu &= at\, v_\nu^* \end{aligned}\right\} (\nu = 1, 2, \ldots, n)$$

$$u_{n+1} = \frac{1}{at} u_{n+1}^*$$

$$v_{n+1} = \frac{1}{at} v_{n+1}^*$$

entsteht aus S' offenbar der Körper S^*

$$|u_{n+1}^* + i v_{n+1}^*| < a$$

$$|u_\nu^* + i v_\nu^*| t + \frac{2^{n+1} n^n |u_{n+1}^* + i v_{n+1}^*|}{a (n+1)^{n+1}} < 1 \text{ für } \frac{|u_{n+1}^* + i v_{n+1}^*|}{a} \leqq \left(\frac{n+1}{2n}\right)^n$$

$$\frac{|u_{n+1}^* + i v_{n+1}^*|}{a} \{ |u_\nu^* + i v_\nu^*| t + 1 \}^n < 1 \text{ für } \left(\frac{n+1}{2n}\right)^n < \frac{|u_{n+1}^* + i v_{n+1}^*|}{a} < 1$$

$$(\nu = 1, 2, \ldots, n)$$

dessen Volumen V^* von uns in einer vorigen Arbeit auf

$$V^* = \frac{a^2}{t^{2n}} \left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} \frac{1}{\varrho_{n,m}^2}$$

berechnet wurde [4]). Weil die Determinante der Transformation den Wert $D = (at)^{2n-2}$ hat, ist also

$$V = D\, V^* = (at)^{2n-2} \frac{a^2}{t^{2n}} \left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} \frac{1}{\varrho_{n,m}^2} = \left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} \quad . \quad . \quad (10)$$

wegen (9).

---

[4]) Nämlich in der unter [2]) zitierten Arbeit, S. 315, 316. Man beachte, dass das in jener Arbeit auftretende $\gamma_{n,m}$ gleich der n-ten Wurzel der in der vorliegenden Arbeit auftretenden Zahl $\varrho_{n,m}$ ist. Dass a in jener Arbeit einen andren Wert hat als in der vorliegenden Arbeit, beeinflusst die Berechnung nicht.

Nach Satz 2 gibt es nun zu jedem vorgegebenen positiven ε im Raum R'_{2n+2} eine Translation

$$z_\nu = u_\nu + d_\nu, \quad w_\nu = v_\nu + e_\nu \quad (\nu = 1, 2, \ldots, n+1) \quad . \quad . \quad (11)$$

durch die der Körper S' in eine solche Lage versetzt wird, dass die Anzahl der „Gitterpunkte" des R'_{2n+2}, welche zu S' oder zu einer ε-Umgebung eines Randpunktes von S' gehören, grösser als $\frac{Vk}{W}$, also wegen (7), (8) und (10) grösser als

$$\left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} k_1 k_2 \left\{\left(\frac{\sqrt{m}}{\lambda}\right)^{n+1} k_1 k_2\right\}^{-1} = 1$$

ist. Es gibt also wenigstens zwei solche „Gitterpunkte"

$$\left.\begin{array}{l}(z'_1, z'_2, \ldots, z'_n, z'_{n+1}, w'_1, w'_2, \ldots, w'_n, w'_{n+1}), \\ (z''_1, z''_2, \ldots, z''_n, z''_{n+1}, w''_1, w''_2, \ldots, w''_n, w''_{n+1});\end{array}\right\} \quad . \quad . \quad . \quad (12)$$

die ihnen vermöge der Translation (11) entsprechenden Punkte seien mit $(u'_1, u'_2, \ldots, u'_n, u'_{n+1}, v'_1, v'_2, \ldots, v'_n, v'_{n+1})$, $(u''_1, u''_2, \ldots, u''_n, u''_{n+1}, \ldots, v''_1, v''_2, \ldots, v''_n, v''_{n+1})$, und die ihnen vermöge (6) entsprechenden Gitterpunkte des R_{2n+2} mit

$$\left.\begin{array}{l}(x'_1, x'_2, \ldots, x'_n, x'_{n+1}, y'_1, y'_2, \ldots, y'_n, y'_{n+1}), \\ (x''_1, x''_2, \ldots, x''_n, x''_{n+1}, y''_1, y''_2, \ldots, y''_n, y''_{n+1})\end{array}\right\} \quad . \quad . \quad . \quad (13)$$

angedeutet. Wir nehmen jetzt die in der Bemerkung 1 beim Satz 1 genannte Zahl $\delta > 0$ beliebig, aber ohne Beschränkung der Allgemeinheit $\leqq \frac{1}{2}$ an, und wir wählen ein positives ε' mit

$$\left.\begin{array}{l}2(1+\varepsilon') < t \text{ und } \varepsilon' < \frac{1}{100}\,\delta, \text{ also mit} \\ 1 < \left(\frac{1+\varepsilon'}{1-\varepsilon'}\right)^{\frac{1}{n}} \leqq \frac{1+\varepsilon'}{1-\varepsilon'} < 1 + 3\,\varepsilon'.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

Dann ist in der obigen Aussage, wo man offenbar $\varepsilon = \varepsilon(\varepsilon') > 0$ nur hinreichend klein zu wählen hat, folgendes enthalten:

$$|u'_{n+1} + i\,v'_{n+1}| < \frac{1+\varepsilon'}{t}, \quad |u''_{n+1} + i\,v''_{n+1}| < \frac{1+\varepsilon'}{t} \quad . \quad . \quad (15)$$

und entweder

$$\frac{|u'_\nu + i\,v'_\nu|}{a} + \frac{2^{n+1}\,n^n\,|u'_{n+1} + i\,v'_{n+1}|\,t}{(n+1)^{n+1}} < 1 + \varepsilon' \text{ mit}$$

$$|u'_{n+1} + i\,v'_{n+1}|\,t \leqq \left(\frac{n+1}{2\,n}\right)^n (1+\varepsilon'),$$

oder

$$|u'_{n+1}+i\,v'_{n+1}|\,t\left\{\frac{|u'_\nu+i\,v'_\nu|}{a}+1\right\}^n<1+\varepsilon' \text{ mit}$$

$$\left(\frac{n+1}{2n}\right)^n(1-\varepsilon')<|u'_{n+1}+i\,v'_{n+1}|\,t<1+\varepsilon'$$

sowie entweder

$$\frac{|u''_\nu+i\,v''_\nu|}{a}+\frac{2^{n+1}\,n^n\,|u''_{n+1}+i\,v''_{n+1}|\,t}{(n+1)^{n+1}}<1+\varepsilon' \text{ mit}$$

$$|u''_{n+1}+i\,v''_{n+1}|\,t<\left(\frac{n+1}{2n}\right)^n(1+\varepsilon'),$$

oder

$$|u''_{n+1}+i\,v''_{n+1}|\,t\left\{\frac{u''_\nu+i\,v''_\nu}{a}+1\right\}^n<1+\varepsilon' \text{ mit}$$

$$\left(\frac{n+1}{2n}\right)^n(1-\varepsilon')<|u''_{n+1}+i\,v''_{n+1}|\,t<1+\varepsilon'.$$

Wir behaupten nun zunächst, dass es unmöglich ist, dass für alle $\nu=1,2,\ldots,n$

$$u'_\nu=u''_\nu,\quad v'_\nu=v''_\nu$$

gelten würde. Denn dann wäre auch wegen (11) für $\nu=1,2,\ldots,n$

$$z'_\nu=z''_\nu,\quad w'_\nu=w''_\nu,\quad \ldots\ldots\ldots \tag{16}$$

und also wäre wenigstens eine der beiden Ungleichungen

$$z'_{n+1}\neq z''_{n+1},\quad w'_{n+1}\neq w''_{n+1}$$

d.h. wegen (11) mindestens eine der beiden Ungleichungen

$$u'_{n+1}\neq u''_{n+1},\quad v'_{n+1}\neq v''_{n+1}$$

erfüllt, so dass wegen (6) und (16) gelten würde (man beachte (1))

$$|u'_{n+1}-u''_{n+1}+i\,(v'_{n+1}-v''_{n+1})|\geqq min.\left(1,\frac{\sqrt{m}}{\lambda}\right)=1 \text{ für } m\neq 3\,. \tag{17}$$

$$|u'_{n+1}-u''_{n+1}+i\,(v'_{n+1}-v''_{n+1})|\geqq min.\left(1,\sqrt{\frac{3}{4}+\left(\frac{1}{2}\right)^2}\right)=1 \text{ für } m=3 \tag{18}$$

Die Ungleichungen (17) und (18) widersprechen aber der aus (15) hervorgehenden Ungleichung

$$|u'_{n+1}-u''_{n+1}+i\,(v'_{n+1}-v''_{n+1})|<\frac{2\,(1+\varepsilon')}{t}<1$$

(siehe (14)). Es gibt also wenigstens ein $\nu\,(1\leqq\nu\leqq n)$ für das mindestens eine der beiden Ungleichungen

$$u'_\nu\neq u''_\nu,\quad v'_\nu\neq v''_\nu$$

erfüllt ist; hieraus folgert man aber sofort die Richtigkeit der Formeln (17) und (18), jetzt aber für jenes ν, statt für $n+1$. Es gilt also für wenigstens ein ν $(1 \leqq \nu \leqq n)$ die Ungleichung

$$|u'_\nu - u''_\nu + i(v'_\nu - v''_\nu)| \geqq 1. \quad \ldots \ldots \quad (19)$$

Wir setzen jetzt für die Punkte (13):

$$\left.\begin{aligned} &X_\nu = x'_\nu - x''_\nu,\ Y_\nu = y'_\nu - y''_\nu \qquad (\nu = 1, 2, \ldots, n, n+1) \\ &P_\nu = X_\nu + \frac{\chi + i\sqrt{m}}{\lambda} Y_\nu \qquad (\nu = 1, 2, \ldots, n) \\ &Q = X_{n+1} + \frac{\chi + i\sqrt{m}}{\lambda} Y_{n+1} \\ &L = \theta_1 P_1 + \theta_2 P_2 + \ldots + \theta_n P_n - Q \end{aligned}\right\} \quad . \ . \quad (20)$$

Wegen (11), (6) und (20) gilt für $\nu = 1, 2, \ldots, n$

$$\left.\begin{aligned} u'_\nu - u''_\nu + i(v'_\nu - v''_\nu) &= z'_\nu - z''_\nu + i(w'_\nu - w''_\nu) \\ &= X_\nu + \frac{\chi}{\lambda} Y_\nu + i\frac{\sqrt{m}}{\lambda} Y_\nu = P_\nu \end{aligned}\right\} \quad . \ . \quad (21)$$

und

$$\left.\begin{aligned} u'_{n+1} - u''_{n+1} + i(v'_{n+1} - v''_{n+1}) &= z'_{n+1} - z''_{n+1} + i(w'_{n+1} - (w''_{n+1}) \\ &= \sum_{\nu=1}^{n} \left\{ \alpha_\nu X_\nu + \left(\alpha_\nu \frac{\chi}{\lambda} - \beta_\nu \frac{\sqrt{m}}{\lambda}\right) Y_\nu \right\} - X_{n+1} - \frac{\chi}{\lambda} Y_{n+1} \\ &+ i \sum_{\nu=1}^{n} \left\{ \beta_\nu X_\nu + \left(\beta_\nu \frac{\chi}{\lambda} + \alpha_\nu \frac{\sqrt{m}}{\lambda}\right) Y_\nu \right\} - i\frac{\sqrt{m}}{\lambda} Y_{n+1} \\ &= \sum_{\nu=1}^{n} \theta_\nu P_\nu - Q = L \end{aligned}\right\} . \quad (22)$$

Es ist nun $(P_1, P_2, \ldots, P_n, Q)$ ein System ganzer Zahlen aus $K(i\sqrt{m})$, für die wegen (21) und (19) gilt

$$P = \max.(|P_1|, |P_2|, \ldots, |P_n|) \geqq 1.$$

Wir können nun auf Grund der jetzt erreichten Ergebnisse die folgenden Ungleichungen herleiten:

$$\left.\begin{aligned} |L| &< \frac{2+\delta}{t} \\ P &< (2+\delta)\,a \\ P\,|L|^{\frac{1}{n}} &< \frac{a}{t^{\frac{1}{n}}}(1+\delta). \end{aligned}\right\} \quad \ldots \ldots \quad (23)$$

Weil aber die Beweise dieser Ungleichungen auf genau dieselbe Weise geführt werden als die Beweise der entsprechenden Ungleichungen im reellen Fall, welche wir in der unter [1]) zitierten Arbeit in allen Einzelheiten durchgeführt haben [5]), und überdies die geringfügigen Abweichungen, die durch das Auftreten des Komplexen nötig sind, in den entsprechenden Beweisen der analogen Ungleichungen unsrer unter [2]) zitierten Arbeit aufs klarste hervortreten [6]), glauben wir die Herleitung von (23) dem Leser überlassen zu dürfen.

Aus (23) folgt wegen (9) nach der Bemerkung 1 beim Satz 1 aber sofort die Richtigkeit dieses Satzes.

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[5]) Es sind die Ungleichungen (27), (28), (29) jener Arbeit und die Beweise findet man auf den Seiten 71—74.

[6]) Es sind die Ungleichungen (32), (33), (34) und die Beweise findet man auf den Seiten 320—323 jener Arbeit.

Mathematics. — *Integraldarstellungen für* WHITTAKER*sche Funktionen und ihre Produkte.* (Erste Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 1. Ich wiederhole zunächst zwei Formeln, die ich neuerdings [1]) angewendet habe zur Ableitung von Integraldarstellungen für BESSELsche, STRUVEsche und LOMMELsche Funktionen. Ich nehme an, dass der Leser mit den Bezeichnungen vertraut ist und überdies die einfachsten Eigenschaften der Funktion $G_{p,q}^{m,n}(\zeta)$ kennt [2]).

Die erste in Frage stehende Formel ist [3])

$$\left.\begin{aligned} G_{p,q}^{m,n}\left(\zeta\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) &= \frac{1}{\Gamma(\alpha+\beta-b_1-b_2)}\int_1^\infty G_{p,q}^{m,n}\left(\zeta v\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ \alpha,\beta,b_3,\ldots,b_q\end{matrix}\right)\\ &\times {}_2F_1(\alpha-b_1,\alpha-b_2;\alpha+\beta-b_1-b_2;1-v)\,(v-1)^{\alpha+\beta-b_1-b_2-1}\,v^{-\beta}\,dv;\end{aligned}\right\} \quad (1)$$

diese Beziehung gilt unter den Voraussetzungen [4])

$$m \geqq 2,\ 2m+2n-p-q>0, \quad \ldots\ldots \quad (2)$$

$$\zeta \neq 0,\ |\arg\zeta| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi, \quad \ldots\ldots \quad (3)$$

$$\Re(a_j)-1<\Re(b_1),\ \Re(a_j)-1<\Re(b_2) \qquad (j=1,\ldots,n) \quad . \quad (4)$$

und

$$\Re(b_1+b_2)<\Re(\alpha+\beta). \quad \ldots\ldots \quad (5)$$

Die zweite Formel lautet wie folgt [5])

$$G_{p,q}^{m,n}\left(\zeta\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{\Gamma(\alpha-b_1)}\int_1^\infty G_{p,q}^{m,n}\left(\zeta v\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ \alpha,b_2,\ldots,b_q\end{matrix}\right)(v-1)^{\alpha-b_1-1}\,v^{-\alpha}\,dv; \quad (6$$

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1) MEIJER, [20].

2) In Bezug auf die Funktion $G_{p,q}^{m,n}(\zeta)$ vergl. man [**10**], 11; [**20**], 199; [**21**], 82—83.

3) Man vergl. [20], 199, Satz 1.

4) In [20] gebe ich etwas allgemeinere Bedingungen für die Gültigkeit von (1); in der vorliegenden Arbeit aber brauche ich nur die einfachen Voraussetzungen (2), (3), (4) und (5). Desgl. bei (6).

5) [20], 204, Satz 2.

hierin wird

$$m \geqq 1, \; 2m + 2n - p - q > 0,$$

$$\zeta \neq 0, \; |\arg \zeta| < (m + n - \tfrac{1}{2} p - \tfrac{1}{2} q)\pi,$$

$$\Re(a_j) - 1 < \Re(b_1) \qquad (j = 1, \ldots, n)$$

und

$$\Re(b_1) < \Re(a)$$

angenommen.

In der vorliegenden Note werde ich (1) und (6) auch auf WHITTAKERsche Funktionen anwenden; ich werde nämlich mit Hilfe von (1) und (6) Integraldarstellungen ableiten für die WHITTAKERschen Funktionen $W_{k,m}(z)$ und $M_{k,m}(z)$ und für die Produkte $W_{k,m}(z) W_{-k,m}(z)$ und $W_{k,m}(iz) W_{k,m}(-iz)$. Es wird sich zeigen, dass verschiedene Formeln, die ich früher auf andere Weise gefunden habe, jetzt zum Vorschein kommen als Spezialfälle von (1).

Die Funktionen $W_{k,m}(z)$ und $M_{k,m}(z)$ können folgenderweise in die G-Funktion ausgedrückt werden [6]):

$$W_{k,m}(z) = e^{\frac{1}{2}z}\, G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1-k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \quad \ldots\ldots \quad (7)$$

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1+k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \quad . \; . \quad (8)$$

$$M_{k,m}(z) = \frac{e^{\frac{1}{2}z}\,\Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}\, G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 1-k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \quad . \; . \; . \quad (9)$$

$$W_{k,m}(z) = \frac{2^{k-\frac{1}{2}}\, z^{\frac{1}{2}}\, e^{\frac{1}{2}z}}{\sqrt{\pi}}\, G_{2,4}^{4,0}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{3}{4}-\frac{1}{2}k, \frac{1}{4}-\frac{1}{2}k \\ \frac{1}{2}m, -\frac{1}{2}m, \frac{1}{2}+\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m \end{matrix}\right). \quad (10)$$

Für die Produkte $W_{k,m}(z) W_{-k,m}(z)$ und $W_{k,m}(iz) W_{k,m}(-iz)$ gelten die Beziehungen

$$W_{k,m}(z) W_{-k,m}(z) = \frac{z}{2\sqrt{\pi}}\, G_{2,4}^{4,0}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+k, \frac{1}{2}-k \\ 0, \frac{1}{2}, m, -m \end{matrix}\right), \quad . \quad (11)$$

$$W_{k,m}(iz)\, W_{k,m}(-iz) = \frac{z}{2\sqrt{\pi}\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, G_{2,4}^{4,1}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+k, \frac{1}{2}-k \\ 0, \frac{1}{2}, m, -m \end{matrix}\right). \quad (12)$$

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[6]) Für (7), (8), (9), (10), (11) und (12) vergl. man [**10**], 12—14; [**21**], 187 und 194.

Auch die Funktion $K_\nu(z)$ und die LOMMELsche Funktion[7] $S_{\mu,\nu}(z)$ sind Spezialfälle der Funktion $G_{p,q}^{m,n}(\zeta)$; man hat nämlich[8])

$$K_\nu(z) = \tfrac{1}{2}\, G_{0,2}^{2,0}\left(\tfrac{1}{4} z^2 \,\middle|\, \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu\right), \quad \ldots \ldots \quad (13)$$

und

$$S_{\mu,\nu}(z) = \frac{2^{\mu-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)}\, G_{1,3}^{3,1}\left(\tfrac{1}{4} z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{matrix}\right). \quad (14)$$

In den §§ 3—8 benutze ich ferner noch die bekannten Relationen

$$G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1+\tau, \ldots, a_p+\tau \\ b_1+\tau, \ldots, b_q+\tau \end{matrix}\right) = \zeta^\tau\, G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \quad . \; . \quad (15)$$

und

$$G_{p+1,q+1}^{m+1,n}\left(\zeta \,\middle|\, \begin{matrix} a_1, \ldots, a_p, \sigma \\ \sigma, b_1, \ldots, b_q \end{matrix}\right) = G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \quad (n \leqq p;\, m \geqq 1). \quad (16)$$

§ 2. Weil im Folgenden wiederholt zugeordnete LEGENDREsche Funktionen erster Art auftreten, werde ich einige Hilfsformeln über diese Funktionen vorausschicken.

Die zugeordnete LEGENDREsche Funktion erster Art $P_n^l(w)$ wird in der von $w=1$ bis $w=-\infty$ aufgeschnittenen w-Ebene definiert durch [9])

$$P_n^l(w) = \frac{(w+1)^{\frac{1}{2}l}(w-1)^{-\frac{1}{2}l}}{\Gamma(1-l)}\, {}_2F_1\left(-n, 1+n; 1-l; \tfrac{1}{2}-\tfrac{1}{2}w\right) \; . . \; . \quad (17)$$

Ist $-1 < w < 1$, so wird die zugeordnete LEGENDREsche Funktion erster Art $\mathbf{P}_n^l(w)$ definiert durch

$$\mathbf{P}_n^l(w) = \frac{(1+w)^{\frac{1}{2}l}(1-w)^{-\frac{1}{2}l}}{\Gamma(1-l)}\, {}_2F_1\left(-n, 1+n; 1-l; \tfrac{1}{2}-\tfrac{1}{2}w\right). \quad . \quad (18)$$

Für die Funktion $P_n^l(w)$ gilt auch

$$P_n^l(w) = \frac{2^l (w^2-1)^{-\frac{1}{2}l}}{\Gamma(1-l)}\, {}_2F_1\left(-\tfrac{1}{2}n-\tfrac{1}{2}l, \tfrac{1}{2}+\tfrac{1}{2}n-\tfrac{1}{2}l; 1-l; 1-w^2\right). \quad (19)$$

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7) Für die Definition der Funktionen $K_\nu(z)$ und $S_{\mu,\nu}(z)$ vergl. man WATSON, [22], 78 und 347—349.

8) [20], 206, Formeln (36) und (42).

9) Für (17), (18) und (19) vergl. man HOBSON, [5], 188, 227 und 219.

Aus (17) mit $w = \cosh 2t$ geht hervor

$$ {}_2F_1(-n, 1+n; 1-l; -\sinh^2 t) = \Gamma(1-l)\sinh^l t\cosh^{-l} t\, P_n^l(\cosh 2t); \quad (20)$$

ebenso liefert (18) mit $w = \cos 2\varphi$

$$ {}_2F_1(-n, 1+n; 1-l; \sin^2\varphi) = \Gamma(1-l)\sin^l\varphi\cos^{-l}\varphi\, \mathbf{P}_n^l(\cos 2\varphi) \quad . \quad (21)$$

Schliesslich folgt aus (19) mit $w = \cosh t$

$$ {}_2F_1(-\tfrac{1}{2}n - \tfrac{1}{2}l,\ \tfrac{1}{2} + \tfrac{1}{2}n - \tfrac{1}{2}l;\ 1-l; -\sinh^2 t) = 2^{-l}\Gamma(1-l)\sinh^l t\, P_n^l(\cosh t) \quad . \quad (22)$$

Durch Anwendung von

$$ {}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(c-b, a; c; \frac{z}{z-1}\right)$$

auf (20) und (21) findet man [10])

$$ {}_2F_1(1+n, 1+n-l; 1-l; \operatorname{tgh}^2 t) = \Gamma(1-l)\sinh^l t\cosh^{2n-l+2} t\, P_n^l(\cosh 2t) \quad . \quad . \quad (23)$$

und

$$ {}_2F_1(1+n, 1+n-l; 1-l; -\operatorname{tg}^2\varphi) = \Gamma(1-l)\sin^l\varphi\cos^{2n-l+2}\varphi\, \mathbf{P}_n^l(\cos 2\varphi). \quad (24)$$

Nun ist

$$ {}_2F_1(\tfrac{1}{2}-k, -k; \tfrac{1}{2}; w^2) = \tfrac{1}{2}\{(1+w)^{2k} + (1-w)^{2k}\}$$

und

$$ {}_2F_1(\tfrac{1}{2}-k, -k; \tfrac{1}{2}; -w^2) = \tfrac{1}{2}\{(1+iw)^{2k} + (1-iw)^{2k}\};$$

es gilt daher

$$ {}_2F_1(\tfrac{1}{2}-k, -k; \tfrac{1}{2}; \operatorname{tgh}^2 t) = \tfrac{1}{2}\{(1+\operatorname{tgh} t)^{2k} + (1-\operatorname{tgh} t)^{2k}\}$$

$$ = \tfrac{1}{2}\cosh^{-2k} t\{(\cosh t + \sinh t)^{2k} + (\cosh t - \sinh t)^{2k}\} = \cosh^{-2k} t\cosh 2kt$$

und ebenso [11])

$$ {}_2F_1(\tfrac{1}{2}-k, -k; \tfrac{1}{2}; -\operatorname{tg}^2\varphi) = \cos^{-2k}\varphi\cos 2k\varphi \quad . \quad . \quad (25)$$

Mit Rücksicht auf (23) und (24) hat man also [12])

$$ P^{\frac{1}{2}}_{\frac{1}{2}\mu-\frac{1}{2}}(\cosh 2t) = \frac{\cosh\mu t}{\sqrt{\pi}\sinh^{\frac{1}{2}} t\cosh^{\frac{1}{2}} t} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

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10) Man vergl. auch HOBSON, [5], 210, Formel (41).

11) Man vergl. auch GAUSS, [4], 127, Formel (XXII).

12) Die Beziehungen (26) und (27) können auch auf andere Weise abgeleitet werden; man vergl. HOBSON, [5], 286.

und

$$\mathsf{P}^{\frac{1}{2}}_{\frac{1}{2}\mu-\frac{1}{2}}(\cos 2\varphi)=\frac{\cos\mu\varphi}{\sqrt{\pi}\,\sin^{\frac{1}{2}}\varphi\,\cos^{\frac{1}{2}}\varphi}\quad\ldots\ldots\quad(27)$$

§ 3. Durch Anwendung von (1) mit $\alpha=\frac{1}{2}+\lambda+\mu$ und $\beta=\frac{1}{2}+\lambda-\mu$ auf (7) findet man

$$\left.\begin{aligned}W_{k,m}(z)&=\frac{e^{\frac{1}{2}z}}{\Gamma(2\lambda)}\int_1^\infty G^{2,0}_{1,2}\left(zv\,\middle|\,\begin{matrix}1-k\\ \frac{1}{2}+\lambda+\mu,\ \frac{1}{2}+\lambda-\mu\end{matrix}\right)\\ &\times{}_2F_1(\lambda+\mu-m,\lambda+\mu+m;2\lambda;1-v)\,(v-1)^{2\lambda-1}\,v^{\mu-\lambda-\frac{1}{2}}\,dv.\end{aligned}\right\}\quad\ldots\quad(28)$$

Für die hierin vorkommende Funktion $G^{2,0}_{1,2}(zv)$ gilt wegen (15) und (7)

$$G^{2,0}_{1,2}\left(zv\,\middle|\,\begin{matrix}1-k\\ \frac{1}{2}+\lambda+\mu,\ \frac{1}{2}+\lambda-\mu\end{matrix}\right)=(zv)^\lambda\,G^{2,0}_{1,2}\left(zv\,\middle|\,\begin{matrix}1-k-\lambda\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right)$$
$$=(zv)^\lambda\,e^{-\frac{1}{2}zv}\,W_{k+\lambda,\mu}(zv).$$

Die Funktion $W_{k,m}(z)$ besitzt daher mit Rücksicht auf (28) die Integraldarstellung

$$\left.\begin{aligned}W_{k,m}(z)&=\frac{z^\lambda e^{\frac{1}{2}z}}{\Gamma(2\lambda)}\int_1^\infty e^{-\frac{1}{2}zv}\,W_{k+\lambda,\mu}(zv)\\ &\times{}_2F_1(\lambda+\mu-m,\lambda+\mu+m;2\lambda;1-v)\,(v-1)^{2\lambda-1}\,v^{\mu-\frac{1}{2}}\,dv;\end{aligned}\right\}\quad.\quad(29)$$

hierin ist $z\neq 0$, $|\arg z|<\frac{1}{2}\pi$ und $\Re(\lambda)>0$.

Nun gilt bekanntlich

$$W_{\frac{1}{2}+m,m}(\zeta)=W_{\frac{1}{2}+m,-m}(\zeta)=\zeta^{\frac{1}{2}+m}\,e^{-\frac{1}{2}\zeta};\quad\ldots\ldots\quad(30)$$

Formel (29) mit $\mu=\frac{1}{2}-k-\lambda$ und $v=1+\dfrac{u}{z}$ liefert also

$$W_{k,m}(z)=\frac{z^k e^{-\frac{1}{2}z}}{\Gamma(2\lambda)}\int_0^\infty e^{-u}\,{}_2F_1(\tfrac{1}{2}-k-m,\tfrac{1}{2}-k+m;2\lambda;-u/z)\,u^{2\lambda-1}\,du;\quad(31)$$

diese Beziehung tritt auch auf in einigen vorigen Arbeiten [13]) des Verfassers.

Für $\mu=m+\lambda$ geht (29) wegen

$${}_2F_1(a,b;a;w)=(1-w)^{-b}\quad\ldots\ldots\ldots\quad(32)$$

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[13]) [9], 36; [11], 478—479; [17], 1098, 1101 und 141—143. Siehe auch ERDÉLYI, [3], 211.

in

$$W_{k,m}(z) = \frac{z^{\lambda} e^{\frac{1}{2}z}}{\Gamma(2\lambda)} \int_1^{\infty} e^{-\frac{1}{2}zv}\, W_{k+\lambda, m+\lambda}(zv)\,(v-1)^{2\lambda-1}\, v^{-m-\lambda-\frac{1}{2}}\, dv \qquad (33)$$

über; auch diese Relation habe ich schon früher [14]) abgeleitet. Der Spezialfall von (33) mit $\lambda = \frac{1}{4} - \frac{1}{2}k - \frac{1}{2}m$ und $v = 1 + \frac{u}{z}$ ergibt die bekannte Formel

$$W_{k,m}(z) = \frac{z^{k} e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-k-m)} \int_0^{\infty} e^{-u} \left(1 + \frac{u}{z}\right)^{k-m-\frac{1}{2}} u^{-k-m-\frac{1}{2}}\, du;$$

diese Beziehung kommt auch zum Vorschein, wenn man $\lambda = \frac{1}{4} - \frac{1}{2}k - \frac{1}{2}m$ setzt in (31).

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[14]) [17], 143.

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(Communicated at the meeting of March 29, 1941.)

§ 11. Wegen (75) und (101) hat man

$$G_{2,4}^{2,2}\left(w^2 \,\middle|\, \begin{matrix} k, \frac{1}{2}+k \\ m, -m, 2k+m, 2k-m \end{matrix}\right)$$

$$= w^{2k}\, G_{2,4}^{2,2}\left(w^2 \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ m-k, -m-k, k+m, k-m \end{matrix}\right) = 2\sqrt{\pi}\, w^{2k} I_{-2k}(w)\, K_{2m}(w).$$

Aus (63) mit $\zeta = z^2, v = u^2, \beta = 2k, \varkappa = \frac{1}{2} - k$ und $\lambda = m$ folgt also

$$W_{k,m}(z^2) = \frac{4\sqrt{\pi}\, z^{1+2k}\, e^{-\frac{1}{2}z^2}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)} \int_0^\infty u^{-2k} e^{-\frac{1}{2}u^2} W_{\frac{1}{2}-k,m}(u^2)\, I_{-2k}(zu)\, K_{2m}(zu)\, du; \quad (13$$

diese Beziehung gilt für $z \neq 0$ und $\Re\left(\frac{1}{2} - k \pm m\right) > 0$.

Auf analoge Weise liefert (63) mit $\zeta = z^2, v = u^2, \beta = 1 + 2k, \varkappa = \frac{1}{2} - k$ und $\lambda = m$

$$W_{k,m}(z^2) = \frac{4\sqrt{\pi}\, z^{2+2k}\, e^{-\frac{1}{2}z^2}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)} \int_0^\infty u^{-2k-1} e^{-\frac{1}{2}u^2} W_{\frac{1}{2}-k,m}(u^2)\, I_{-2k-1}(zu)\, K_{2m}(zu)\, du;$$

hierin ist $z \neq 0$ und $\Re(k) \pm \Re(m) < 0$.

Der Spezialfall von (136) mit $k = -m$ ist schon von ERDÉLYI [38]) gefunden worden. Nimmt man $k = \frac{1}{4} + \frac{1}{2}n$ und $m = \frac{1}{4}$ in (136) und in (137) und ersetzt man z durch $\frac{1}{2}z\sqrt{2}$ und u durch $\frac{1}{2}u\sqrt{2}$, so erhält man mit Rücksicht auf (111) und (117)

$$\left.\begin{aligned} D_n(z) = \frac{\sqrt{\pi}\, z^{n+\frac{1}{2}}}{\Gamma(-n)} \int_0^\infty u^{-n-\frac{1}{2}} e^{-\frac{1}{4}(z+u)^2} D_{-n}(u)\, I_{-n-\frac{1}{2}}\left(\tfrac{1}{2}zu\right) du \\ [z \neq 0;\ \Re(n) < 0] \end{aligned}\right\} \tag{138}$$

bezw.

$$\left.\begin{aligned} D_n(z) = \frac{\sqrt{\pi}\, z^{n+\frac{3}{2}}}{\Gamma(-n)} \int_0^\infty u^{-n-\frac{3}{2}} e^{-\frac{1}{4}(z+u)^2} D_{-n}(u)\, I_{-n-\frac{3}{2}}\left(\tfrac{1}{2}zu\right) du \\ [z \neq 0;\ \Re(n) < -1]. \end{aligned}\right\} \tag{139}$$

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38) ERDÉLYI, [8], Formel (14).

Setzt man $k = \beta = 0, \varkappa = \frac{1}{2}, \zeta = 2z^2$ und $v = 2u^2$ in (63), so bekommt man infolge (113) und (102)

$$(z^2) = \frac{\sqrt{2}\,\pi e^{-z^2}}{\sin m\pi} \int_0^\infty e^{-u^2} W_{\frac{1}{4},\lambda}(2u^2) \{ I_{\lambda-m}(2zu) I_{-\lambda-m}(2zu) - I_{m-\lambda}(2zu) I_{m+\lambda}(2zu) \} du; \quad (140)$$

hierin ist $z \neq 0$, $|\Re(m)| < 1$ und λ beliebig mit $|\Re(\lambda)| + |\Re(m)| < 1$.

Der Spezialfall mit $\lambda = 0$ ergibt wegen (116)

$$K_m(z^2) = \frac{2\pi e^{-z^2}}{\sin m\pi} \int_0^\infty u\, e^{-2u^2} \{ I_{-m}^2(2zu) - I_m^2(2zu) \} du; \quad (141)$$

diese Beziehung habe ich früher [39]) auf andre Weise gefunden.

Ich werde jetzt einige mit (136) und (137) verwandten Relationen ableiten. Dazu substituiere ich

$\zeta = z^2,\ v = u^2,\ \psi = 2\varphi,\ \beta = 2k,\ \varkappa = -\frac{1}{2} + k$ und $\lambda = m$ in (66),
$\zeta = z^2,\ v = u^2,\ \psi = 2\varphi,\ \beta = 1 + 2k,\ \varkappa = -\frac{1}{2} + k$ und $\lambda = m$ in (66),
$\zeta = z^2,\ v = u^2,\ \beta = -2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = m$ in (68),
$\zeta = z^2,\ v = u^2,\ \beta = 1 - 2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = m$ in (68),
$\zeta = z^2,\ v = u^2,\ \beta = -2k,\ \varkappa = -\frac{1}{2} - k$ und $\lambda = m$ in (69),
$\zeta = z^2,\ v = u^2,\ \beta = 1 - 2k,\ \varkappa = -\frac{1}{2} - k$ und $\lambda = m$ in (69),
$\zeta = z^2,\ v = u^2,\ \beta = -2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = m$ in (70),
$\zeta = z^2,\ v = u^2,\ \beta = 1 - 2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = m$ in (70),
$\zeta = z^2,\ v = u^2,\ \beta = -2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = -m$ in (71),
$\zeta = z^2,\ v = u^2,\ \beta = 1 - 2k,\ \varkappa = \frac{1}{2} + k$ und $\lambda = -m$ in (71).

Diese Substitutionen ergeben successive mit Rücksicht auf (104), (103), (99), (100), (96), (97) und (98)

$$W_{k,m}(z^2) = \frac{z^{1+2k} e^{-\frac{1}{2}z^2} \Gamma(1-k+m)\, \Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k-m)}{2\sqrt{\pi}\, i\, \Gamma(1+2m)} \times \int_B u^{-2k} e^{\frac{1}{2}u^2} M_{-\frac{1}{2}+k,m}(u^2)\, V_{2k,2m}(zu)\, du$$

(wo $z \neq 0$ und $|\arg z| < \frac{3}{4}\pi$),

$$W_{k,m}(z^2) = \frac{z^{2+2k} e^{-\frac{1}{2}z^2} \Gamma(1-k+m)\, \Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k-m)}{2\sqrt{\pi}\, i\, \Gamma(1+2m)} \times \int_B u^{-2k-1} e^{\frac{1}{2}u^2} M_{-\frac{1}{2}+k,m}(u^2)\, U_{2k+1,2m}(zu)\, du$$

(wo $z \neq 0$ und $|\arg z| < \frac{3}{4}\pi$),

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39) MEIJER, [11], Formel (19).

$$W_{k,m}(z^2)=\frac{\sqrt{\pi}\, i\, z^{1-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+k+m)}{2\,\Gamma(1+2m)}$$

$$\times\int_0^{\infty e^{-i\arg z}} u^{2k}\, e^{-\frac{1}{2}u^2} M_{\frac{1}{2}+k,m}(u^2)\,\{H^{(1)}_{2k}(zu)\,H^{(1)}_{2m}(zu)-H^{(2)}_{2k}(zu)\,H^{(2)}_{2m}(zu)\}\,du$$

(wo $z\neq 0$, $|\arg z|<\frac{1}{4}\pi$, $\Re(\frac{1}{2}+m)>0$, $\Re(\frac{1}{2}+k)>0$ und $\Re(\frac{1}{2}+k+m)>0$),

$$W_{k,m}(z^2)=\frac{\sqrt{\pi}\, i\, z^{2-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+k+m)}{2\,\Gamma(1+2m)}$$

$$\times\int_0^{\infty e^{-i\arg z}} u^{2k-1}\, e^{-\frac{1}{2}u^2} M_{\frac{1}{2}+k,m}(u^2)\,\{H^{(1)}_{2k-1}(zu)\,H^{(1)}_{2m}(zu)-H^{(2)}_{2k-1}(zu)\,H^{(2)}_{2m}(zu)\}\,du$$

(wo $z\neq 0$, $|\arg z|<\frac{1}{4}\pi$, $\Re(\frac{1}{2}+m)>0$, $\Re(k)>0$ und $\Re(k+m)>0$),

$$W_{k,m}(z^2)=\frac{2\,z^{1-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+k+m)}{\pi^{\frac{3}{2}}\, i\,\Gamma(1+2m)}\int_C u^{2k}\, e^{\frac{1}{2}u^2} M_{-\frac{1}{2}-k,m}(u^2)\,K_{2k}(zu)\,K_{2m}(zu)\,du$$

(wo $z\neq 0$ und $|\arg z|<\frac{1}{4}\pi$),

$$W_{k,m}(z^2)=\frac{2\,z^{2-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+k+m)}{\pi^{\frac{3}{2}}\, i\,\Gamma(1+2m)}\int_C u^{2k-1}\, e^{\frac{1}{2}u^2} M_{-\frac{1}{2}-k,m}(u^2)\,K_{2k-1}(zu)\,K_{2m}(zu)\,du$$

(wo $z\neq 0$ und $|\arg z|<\frac{1}{4}\pi$),

$$M_{k,m}(z^2)=\frac{2\sqrt{\pi}\, z^{1-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}\int_0^\infty u^{2k}\, e^{-\frac{1}{2}u^2}\, W_{\frac{1}{2}+k,m}(u^2)\,J_{2k}(zu)\,J_{2m}(zu)\,du$$

(wo $z\neq 0$, $\Re(\frac{1}{2}+k)>0$ und $\Re(\frac{1}{2}+k+m)>0$),

$$M_{k,m}(z^2)=\frac{2\sqrt{\pi}\, z^{2-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}\int_0^\infty u^{2k-1}\, e^{-\frac{1}{2}u^2}\, W_{\frac{1}{2}+k,m}(u^2)\,J_{2k-1}(zu)\,J_{2m}(zu)\,d$$

(wo $z\neq 0$, $\Re(k)>0$ und $\Re(k+m)>0$),

$$M_{k,m}(z^2)=\frac{z^{1-2k}\, e^{\frac{1}{2}z^2}\,\Gamma(1+2m)\,\Gamma(1+k-m)\,\Gamma(\frac{1}{2}-k-m)}{\sqrt{\pi}\,\Gamma(1-2m)}$$

$$\times\int_0^{\infty e^{-i\arg z}} u^{2k}\, e^{-\frac{1}{2}u^2} M_{\frac{1}{2}+k,-m}(u^2)\,\{J_{2k}(zu)\,J_{2m}(zu)+J_{-2k}(zu)\,J_{-2m}(zu)\}\,du$$

(wo $z\neq 0$, $|\arg z|<\frac{1}{4}\pi$, $\Re(\frac{1}{2}-m)>0$ und $\Re(\frac{1}{2}+k)>0$),

$$M_{k,m}(z^2) = \frac{z^{2-2k}\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)\, \Gamma(1+k-m)\, \Gamma(\frac{1}{2}-k-m)}{\sqrt{\pi}\, \Gamma(1-2m)}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{2k-1}\, e^{-\frac{1}{2}u^2} M_{\frac{1}{2}+k,-m}(u^2) \{J_{2k-1}(zu)\, J_{2m}(zu) - J_{1-2k}(zu)\, J_{-2m}(zu)\}\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$, $\Re(\frac{1}{2}-m) > 0$ und $\Re(k) > 0$).

Ich gebe jetzt einige mit (140) verwandten Beziehungen. Dazu substituiere ich $\zeta = 2z^2$, $v = 2u^2$ und $k = \beta = 0$ in (62), (66), (67), (68), (69) und (70); überdies setze ich noch $\varkappa = -\frac{1}{2}$ in (62), (66) [40]) und (69) und $\varkappa = \frac{1}{2}$ in (67), (68) und (70). Ich erhalte dann mit Rücksicht auf (113), (112), (97), (104), (98), (99), (100) und (96)

$$K_m(z^2) = \sqrt{2}\, e^{-z^2}\, \Gamma(1+\lambda)\, \Gamma(1-\lambda) \int_0^{\infty e^{-i \arg z}} e^{u^2}\, W_{-\frac{1}{2},\lambda}(2u^2)$$

$$\times \{J_{m-\lambda}(2zu)\, J_{m+\lambda}(2zu) + J_{\lambda-m}(2zu)\, J_{-\lambda-m}(2zu)\}\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$, $|\Re(m)| < 1$ und λ beliebig mit $|\Re(\lambda)| + |\Re(m)| < 1$),

$$K_m(z^2) = \frac{\sqrt{\pi}\, e^{-z^2}\, \Gamma(\frac{1}{2}-\lambda)}{2^{2\lambda+\frac{1}{2}}\, i} \int_B e^{u^2} M_{-\frac{1}{2},\lambda}(2u^2)\; V_{m+\lambda,m-\lambda}(2zu)\, du \qquad (142)$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$ und λ beliebig),

$$\left.\begin{aligned} K_m(z^2) &= \frac{\sqrt{2}\, \pi\, e^{z^2}}{\sin m\pi} \int_0^{\infty} e^{-u^2}\, W_{\frac{1}{2},\lambda}(2u^2) \\ &\times \{J_{\lambda-m}(2zu)\, J_{-\lambda-m}(2zu) - J_{m-\lambda}(2zu)\, J_{m+\lambda}(2zu)\}\, du \end{aligned}\right\} . \qquad (143)$$

(wo $z \neq 0$, $|\Re(m)| < 1$ und λ beliebig mit $|\Re(\lambda)| + |\Re(m)| < 1$),

$$K_m(z^2) = \frac{\pi^{\frac{3}{2}}\, i\, e^{z^2}}{2^{2\lambda+\frac{1}{2}}\, \Gamma(\frac{1}{2}+\lambda)} \int_0^{\infty e^{-i \arg z}} e^{-u^2} M_{\frac{1}{2},\lambda}(2u^2)$$

$$\times \{H^{(1)}_{\lambda+m}(2zu)\, H^{(1)}_{\lambda-m}(2zu) - H^{(2)}_{\lambda+m}(2zu)\, H^{(2)}_{\lambda-m}(2zu)\}\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und λ beliebig mit $\Re(\frac{1}{2}+\lambda) > 0$ und $\Re(1 \pm m + \lambda) > 0$),

$$K_m(z^2) = \frac{e^{z^2}}{2^{2\lambda-\frac{3}{2}}\sqrt{\pi}\, i\, \Gamma(\frac{1}{2}+\lambda)} \int_C e^{u^2} M_{-\frac{1}{2},\lambda}(2u^2)\, K_{m+\lambda}(2zu)\, K_{m-\lambda}(2zu)\, du \quad (144)$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und λ beliebig),

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40) Ich setze ferner $\psi = 2\varphi$ in (66).

$$I_m(z^2) = 2^{\frac{3}{2}} e^{z^2} \int_0^\infty e^{-u^2} W_{\frac{1}{2},\lambda}(2u^2) J_{m+\lambda}(2zu) J_{m-\lambda}(2zu)\, du \qquad (145)$$

(wo $z \neq 0$, $\Re(1+m) > 0$ und λ beliebig mit $\Re(1+m\pm\lambda) > 0$).

Nimmt man $\lambda = 0$ in (143) und in (145), so findet man infolge (116)

$$K_m(z^2) = \frac{2\pi e^{z^2}}{\sin m\pi} \int_0^\infty u\, e^{-2u^2} \{J_{-m}^2(2zu) - J_m^2(2zu)\}\, du \quad . \quad (146)$$

(wo $z \neq 0$ und $|\Re(m)| < 1$)

bezw.

$$I_m(z^2) = 4 e^{z^2} \int_0^\infty u\, e^{-2u^2} J_m^2(2zu)\, du \text{ [41]} \quad . \quad . \quad . \quad . \quad (147)$$

(wo $z \neq 0$ und $\Re(1+m) > 0$).

Die Spezialfälle mit $\lambda = 0$ von (142) und (144) ergeben wegen (115) und (79) die für jedes $z \neq 0$ gültigen Beziehungen

$$K_m(z^2) = \frac{\pi\, e^{-z^2}}{i} \int_E u\, e^{2u^2} H_m^{(1)}(2zu) H_m^{(2)}(2zu)\, du \quad . \quad . \quad (148)$$

und

$$K_m(z^2) = \frac{4 e^{z^2}}{\pi i} \int_E u\, e^{2u^2} K_m^2(2zu)\, du \quad . \quad . \quad . \quad . \quad . \quad (149)$$

Man bekommt Erweiterungen von (148) und (149), wenn man in (66) und in (69)

$$\zeta = 2z^2,\ v = 2u^2,\ k = 0,\ \beta = -\lambda \text{ und } \varkappa = -\tfrac{1}{2} + \lambda \quad . \quad . \quad (150)$$

und in (66) überdies noch $\psi = 2\varphi$ setzt. Diese Substitutionen liefern nämlich infolge (113), (95) und (92)

$$K_m(z^2) = \frac{2^{\lambda-\frac{1}{2}}\pi e^{-z^2}}{i\,\Gamma(1+2\lambda)} \int_B u^{2\lambda} e^{u^2} M_{-\frac{1}{2}+\lambda,\lambda}(2u^2) H_m^{(1)}(2zu) H_m^{(2)}(2zu)\, du \qquad (151)$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$ und λ beliebig mit $\Re(\frac{1}{4} - \lambda) > 0$)

bezw.

$$K_m(z^2) = \frac{2^{\lambda+\frac{3}{2}} e^{z^2}}{\pi i\,\Gamma(1+2\lambda)} \int_C u^{2\lambda} e^{u^2} M_{-\frac{1}{2}+\lambda,\lambda}(2u^2) K_m^2(2zu)\, du \quad . \quad (152)$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und λ beliebig mit $\Re(\frac{1}{2} - \lambda) > 0$).

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[41]) Formel (147) war schon bekannt; man vergl. WATSON, [26], 395, Formel (1). Siehe auch [11], Formel (18).

Man sieht sofort ein, dass (151) und (152) Erweiterungen bezw. von (148) und (149) sind; denn die Spezialfälle mit $\lambda = 0$ von (151) und (152) können wegen (115) in (148) bezw. (149) übergeführt werden.

Der Ansatz (150) in (72) ergibt mit Rücksicht auf (112) und (93)

$$I_m(z^2) = \frac{2^{\lambda+\frac{3}{2}} e^{z^2}}{\pi i \Gamma(1+2\lambda)} \int_C u^{2\lambda} e^{u^2} M_{-\frac{1}{4}+\lambda,\lambda}(2u^2) I_m(2zu) K_m(2zu)\, du;$$

hierin ist $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und λ beliebig mit $\Re(\frac{1}{4} - \lambda) > 0$.

Für $\lambda = 0$ bekommt man die für alle Werte von $z \neq 0$ geltende Relation

$$I_m(z^2) = \frac{4 e^{z^2}}{\pi i} \int_E u\, e^{2u^2} I_m(2zu) K_m(2zu)\, du \quad . \quad . \quad . \quad . \quad (153)$$

Aus (112) und (58) geht hervor

$$e^{z^2} I_m(z^2) = \frac{\sqrt{\pi}}{\cos m\pi} G_{1,2}^{1,0}\left(2z^2 \middle| \begin{matrix} \frac{1}{2} \\ m, -m \end{matrix}\right). \quad . \quad . \quad . \quad (154)$$

Nun folgt aus (50), mit $\omega = 2z^2$, $\eta = 1$, $x = 2u^2$ und $b = 0$ angewendet, falls $z \neq 0$ ist,

$$\frac{2}{\pi i} \int_E u\, e^{2u^2} G_{1,3}^{2,0}\left(4z^2u^2 \middle| \begin{matrix} \frac{1}{2} \\ 0, m, -m \end{matrix}\right) du$$

$$= G_{2,3}^{2,0}\left(2z^2 \middle| \begin{matrix} \frac{1}{2}, 0 \\ 0, m, -m \end{matrix}\right) = G_{1,2}^{1,0}\left(2z^2 \middle| \begin{matrix} \frac{1}{2} \\ m, -m \end{matrix}\right)$$

wegen (73).

Mit Rücksicht auf (154) und (90) gilt also

$$I_m(z^2) = \frac{2i e^{-z^2}}{\cos m\pi} \int_E u\, e^{2u^2} J_m(2zu) Y_m(2zu)\, du.$$

Die Umkehrformeln von (148), (149) und (153) für die LAPLACE-Transformation waren bekannt [42]).

§ 12. Ich substituiere jetzt

$\beta = \lambda = \frac{1}{4} + \frac{1}{2}k$ und $\varkappa = -\frac{3}{4} + \frac{1}{2}k$ in (62),
$\beta = \lambda = \frac{1}{4} + \frac{1}{2}k$ und $\varkappa = \frac{3}{4} - \frac{1}{2}k$ in (63),
$\beta = \frac{1}{4} + \frac{1}{2}k$, $\lambda = -\frac{1}{4} - \frac{1}{2}k$ und $\varkappa = \frac{3}{4} - \frac{1}{2}k$ in (64),
$\beta = \lambda = \frac{1}{4} - \frac{1}{2}k$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k$ in (68),
$\beta = \lambda = \frac{1}{4} - \frac{1}{2}k$ und $\varkappa = -\frac{3}{4} - \frac{1}{2}k$ in (69),
$\beta = \frac{1}{4} - \frac{1}{2}k$, $\lambda = -\frac{1}{4} + \frac{1}{2}k$ und $\varkappa = -\frac{3}{4} - \frac{1}{2}k$ in (69),
$\beta = \lambda = \frac{1}{4} - \frac{1}{2}k$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k$ in (70),
$\beta = \frac{1}{4} - \frac{1}{2}k$, $\lambda = -\frac{1}{4} + \frac{1}{2}k$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k$ in (71);

42) Man vergl. WATSON, [26], 439.

ausserdem setze ich immer $\zeta = z^2$ und $v = u^2$ und ferner noch $\psi = 2\varphi$ in (64). Die resultierenden Beziehungen lauten mit Rücksicht auf (91), (94), (92), (88) und (90)

$$W_{k,m}(z^2) = \frac{\pi z e^{-\frac{1}{2}z^2} \Gamma(1-k)}{2\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin m\pi}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{-k-\frac{1}{2}} e^{\frac{1}{2}u^2} W_{-\frac{3}{4}+\frac{1}{2}k, \frac{1}{4}+\frac{1}{2}k}(u^2) \{J^2_{-m}(zu) - J^2_m(zu)\}\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$, $\Re(1 \pm m) > 0$ und $\Re(\frac{1}{2}-k \pm m) > 0$),

$$\left.\begin{aligned} W_{k,m}(z^2) &= \frac{2\pi^{\frac{3}{2}} z e^{-\frac{1}{2}z^2}}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin 2m\pi} \\ &\times \int_0^{\infty} u^{-k-\frac{1}{2}} e^{-\frac{1}{2}u^2} W_{\frac{3}{4}-\frac{1}{2}k, \frac{1}{4}+\frac{1}{2}k}(u^2) \{I^2_{-m}(zu) - I^2_m(zu)\}\, du \end{aligned}\right\} \quad (155)$$

(wo $z \neq 0$, $\Re(1 \pm m) > 0$ und $\Re(\frac{1}{2}-k \pm m) > 0$),

$$W_{k,m}(z^2) = \frac{4 z e^{-\frac{1}{2}z^2} \Gamma(1-k)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-k)}$$

$$\times \int_0^{\infty e^{i\varphi}} u^{-k-\frac{1}{2}} e^{-\frac{1}{2}u^2} M_{\frac{3}{4}-\frac{1}{2}k, -\frac{1}{4}-\frac{1}{2}k}(u^2) K^2_m(zu)\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$ und $\Re(\frac{1}{2}-k \pm m) > 0$),

$$W_{k,m}(z^2) = \frac{\pi z e^{\frac{1}{2}z^2}}{2\Gamma(\frac{3}{2}-k)\sin m\pi} \int_0^{\infty e^{-i\arg z}} u^{k-\frac{1}{2}} e^{-\frac{1}{2}u^2} M_{\frac{1}{4}+\frac{1}{2}k, \frac{1}{4}-\frac{1}{2}k}(u^2) \{J^2_{-m}(zu) - J^2_m(zu)\}\, du \quad (1$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und $\Re(1 \pm m) > 0$),

$$W_{k,m}(z^2) = \frac{z e^{\frac{1}{2}z^2}}{\pi i \Gamma(\frac{3}{2}-k)} \int_C u^{k-\frac{1}{2}} e^{\frac{1}{2}u^2} M_{-\frac{1}{4}-\frac{1}{2}k, \frac{1}{4}-\frac{1}{2}k}(u^2) K^2_m(zu)\, du \quad . \quad (157)$$

(wo $z \neq 0$ und $|\arg z| < \frac{1}{4}\pi$),

$$W_{k,m}(z^2) = \frac{2 z e^{\frac{1}{2}z^2} \Gamma(1+k)}{\pi^{\frac{3}{2}} i \Gamma(\frac{1}{2}+k)} \int_C u^{k-\frac{1}{2}} e^{\frac{1}{2}u^2} M_{-\frac{1}{4}-\frac{1}{2}k, -\frac{1}{4}+\frac{1}{2}k}(u^2) K^2_m(zu)\, du$$

(wo $z \neq 0$ und $|\arg z| < \frac{1}{4}\pi$),

$$M_{k,m}(z^2) = \frac{2\sqrt{\pi}\, z e^{\frac{1}{2}z^2} \Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)} \int_0^{\infty} u^{k-\frac{1}{2}} e^{-\frac{1}{2}u^2} W_{\frac{1}{4}+\frac{1}{2}k, \frac{1}{4}-\frac{1}{2}k}(u^2) J^2_m(zu)\, du \quad (158)$$

(wo $z \neq 0$, $\Re(1+m) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$),

$$M_{k,m}(z^2) = -\frac{2\sqrt{\pi}\, z\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)\, \Gamma(1+k)}{\Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k)}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{k-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, M_{\frac{3}{4}+\frac{1}{2}k,\, -\frac{1}{4}+\frac{1}{2}k}(u^2)\, J_m(zu)\, Y_m(zu)\, du\,.$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$, $\Re(\frac{1}{2}+k) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$).

Die Relationen (155), (156), (157) und (158) sind Erweiterungen bezw. von (141), (146), (149) und (147); denn die Spezialfälle mit $k=0$ von (155), (156), (157) und (158) können mit Hilfe von (113), (112), (116), (114) und (115) in (141), (146), (149) und (147) übergeführt werden.

§ 13. Jetzt setze ich

$\beta = \frac{1}{4} + \frac{1}{2}k - \frac{3}{2}m$, $\lambda = \frac{1}{4} + \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = -\frac{3}{4} + \frac{1}{2}k - \frac{1}{2}m$ in (62),
$\beta = \frac{1}{4} + \frac{1}{2}k - \frac{3}{2}m$, $\lambda = \frac{1}{4} + \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = \frac{3}{4} - \frac{1}{2}k + \frac{1}{2}m$ in (63),
$\beta = \frac{1}{4} + \frac{1}{2}k - \frac{3}{2}m$, $\lambda = -\frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$ und $\varkappa = \frac{3}{4} - \frac{1}{2}k + \frac{1}{2}m$ in (64),
$\beta = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$, $\lambda = \frac{1}{4} - \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k + \frac{1}{2}m$ in (68),
$\beta = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$, $\lambda = \frac{1}{4} - \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = -\frac{3}{4} - \frac{1}{2}k - \frac{1}{2}m$ in (69),
$\beta = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$, $\lambda = -\frac{1}{4} + \frac{1}{2}k - \frac{3}{2}m$ und $\varkappa = -\frac{3}{4} - \frac{1}{2}k - \frac{1}{2}m$ in (69),
$\beta = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$, $\lambda = \frac{1}{4} - \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k + \frac{1}{2}m$ in (70),
$\beta = \frac{1}{4} - \frac{1}{2}k + \frac{3}{2}m$, $\lambda = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k - \frac{1}{2}m$ in (70),
$\beta = \frac{1}{4} - \frac{1}{2}k + \frac{3}{2}m$, $\lambda = -\frac{1}{4} + \frac{1}{2}k + \frac{3}{2}m$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k - \frac{1}{2}m$ in (71),
$\beta = \frac{1}{4} - \frac{1}{2}k - \frac{3}{2}m$, $\lambda = -\frac{1}{4} + \frac{1}{2}k - \frac{3}{2}m$ und $\varkappa = \frac{3}{4} + \frac{1}{2}k + \frac{1}{2}m$ in (71);

ferner ersetze ich immer wieder ζ durch z^2 und v durch u^2 und in (64) überdies noch ψ durch 2φ.

Die erzielten Relationen lauten successive infolge (90) [43]), (93), (92), (88), (89) und (91)

$$W_{k,m}(z^2) = -\frac{2\sqrt{\pi}\, z^{1-2m}\, e^{-\frac{1}{2}z^2}\, \Gamma(\frac{3}{2}+2m)\, \Gamma(1-k-m)}{\Gamma(\frac{1}{2}-k+m)\, \Gamma(\frac{1}{2}-k-m)}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{-k+m-\frac{1}{2}}\, e^{\frac{1}{2}u^2}\, W_{-\frac{3}{4}+\frac{1}{2}k-\frac{1}{2}m,\, \frac{1}{4}+\frac{1}{2}k+\frac{3}{2}m}(u^2)\, J_{2m}(zu)\, Y_{2m}(zu)\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$, $\Re(\frac{1}{4}+m) > 0$ und $\Re(\frac{1}{2}-k \pm m) > 0$),

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[43]) Wegen (74), (75) und (90) gilt

$$G^{2,1}_{2,4}\left(w^2 \,\middle|\, \begin{matrix} \frac{1}{2}+k, \frac{1}{2}-m \\ m, -m, \frac{1}{2}+k, -3m \end{matrix}\right) = G^{2,0}_{1,3}\left(w^2 \,\middle|\, \begin{matrix} \frac{1}{2}-m \\ m, -m, -3m \end{matrix}\right)$$

$$= w^{-2m}\, G^{2,0}_{1,3}\left(w^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 2m, 0, -2m \end{matrix}\right) = -\sqrt{\pi}\, w^{-2m}\, J_{2m}(w)\, Y_{2m}(w).$$

$$W_{k,m}(z^2) = \frac{4\sqrt{\pi}\, z^{1-2m}\, e^{-\frac{1}{2}z^2}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$

$$\times \int_0^\infty u^{-k+m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, W_{\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}m,\,\frac{1}{4}+\frac{1}{2}k+\frac{3}{2}m}(u^2)\, I_{2m}(zu)\, K_{2m}(zu)\, du$$

(wo $z \neq 0$, $\Re(\frac{1}{4}+m) > 0$ und $\Re(\frac{1}{2}-k \pm m) > 0$),

$$\left.\begin{aligned} W_{k,m}(z^2) &= \frac{4\, z^{1-2m}\, e^{-\frac{1}{2}z^2}\, \Gamma(1-k-m)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)\,\Gamma(\frac{1}{2}-k-3m)} \\ &\times \int_0^{\infty e^{i\varphi}} u^{-k+m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, M_{\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}m,\,-\frac{1}{4}-\frac{1}{2}k-\frac{3}{2}m}(u^2)\, K_{2m}^2(zu)\, du \end{aligned}\right\} \quad (159)$$

(wo $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$, $\Re(\frac{1}{2}-k-3m) > 0$ und $\Re(\frac{1}{2}-k+m) > 0$),

$$W_{k,m}(z^2) = -\frac{2\sqrt{\pi}\, z^{1-2m}\, e^{\frac{1}{2}z^2}\, \Gamma(\frac{3}{2}+2m)}{\Gamma(\frac{3}{2}-k+3m)}$$

$$\times \int_0^{\infty e^{-i\arg z}} u^{k+m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, M_{\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}m,\,\frac{1}{4}-\frac{1}{2}k+\frac{3}{2}m}(u^2)\, J_{2m}(zu)\, Y_{2m}(zu)\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$ und $\Re(\frac{1}{4}+m) > 0$),

$$W_{k,m}(z^2) = \frac{2\, z^{1-2m}\, e^{\frac{1}{2}z^2}\, \Gamma(\frac{3}{2}+2m)}{\pi^{\frac{3}{2}}\, i\, \Gamma(\frac{3}{2}-k+3m)}$$

$$\times \int_C u^{k+m-\frac{1}{2}}\, e^{\frac{1}{2}u^2}\, M_{-\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}m,\,\frac{1}{4}-\frac{1}{2}k+\frac{3}{2}m}(u^2)\, K_{2m}^2(zu)\, du$$

(wo $z \neq 0$ und $|\arg z| < \frac{1}{4}\pi$),

$$\left.\begin{aligned} W_{k,m}(z^2) &= \frac{2\, z^{1-2m}\, e^{\frac{1}{2}z^2}\, \Gamma(1+k-m)}{\pi^{\frac{3}{2}}\, i\, \Gamma(\frac{1}{2}+k-3m)} \\ &\times \int_C u^{k+m-\frac{1}{2}}\, e^{\frac{1}{2}u^2}\, M_{-\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}m,\,-\frac{1}{4}+\frac{1}{2}k-\frac{3}{2}m}(u^2)\, K_{2m}^2(zu)\, du \end{aligned}\right\} \quad (160)$$

(wo $z \neq 0$ und $|\arg z| < \frac{1}{4}\pi$),

$$M_{k,m}(z^2) = \frac{2\sqrt{\pi}\, z^{1-2m}\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}$$

$$\times \int_0^\infty u^{k+m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, W_{\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}m,\,\frac{1}{4}-\frac{1}{2}k+\frac{3}{2}m}(u^2)\, J_{2m}^2(zu)\, du$$

(wo $z \neq 0$, $\Re(\frac{1}{4}+m) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$),

$$M_{k,m}(z^2) = \frac{2\sqrt{\pi}\, z^{1+2m}\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}$$

$$\times \int_0^\infty u^{k-m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, W_{\frac{3}{4}+\frac{1}{2}k-\frac{1}{2}m,\frac{1}{4}-\frac{1}{2}k-\frac{3}{2}m}(u^2)\, J_{2m}(zu)\, J_{-2m}(zu)\, du$$

(wo $z \neq 0$, $\Re(\frac{1}{2}-m) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$),

$$M_{k,m}(z^2) = -\frac{2\sqrt{\pi}\, z^{1+2m}\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)\, \Gamma(1+k+m)}{\Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k+3m)}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{k-m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, M_{\frac{3}{4}+\frac{1}{2}k-\frac{1}{2}m,-\frac{1}{4}+\frac{1}{2}k+\frac{3}{2}m}(u^2)\, J_{2m}(zu)\, Y_{2m}(zu)\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$, $\Re(\frac{1}{2}+k+3m) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$),

$$M_{k,m}(z^2) = \frac{\sqrt{\pi}\, z^{1-2m}\, e^{\frac{1}{2}z^2}\, \Gamma(1+2m)\, \Gamma(1+k-m)}{\Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k-3m)\sin 2m\pi}$$

$$\times \int_0^{\infty e^{-i \arg z}} u^{k+m-\frac{1}{2}}\, e^{-\frac{1}{2}u^2}\, M_{\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}m,-\frac{1}{4}+\frac{1}{2}k-\frac{3}{2}m}(u^2)\, \{J_{-2m}^2(zu) - J_{2m}^2(zu)\}\, du$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{4}\pi$, $\Re(\frac{1}{2}+k-3m) > 0$ und $\Re(\frac{1}{2}+k+m) > 0$).

Formel (159) ist eine Verallgemeinerung von (120); (159) mit $k = \frac{1}{4} + \frac{1}{2}n$ und $m = -\frac{1}{4}$ geht nämlich nach einiger Reduktion mittels (111), (114) und (117) in (120) über.

Der Spezialfall mit $k = \frac{1}{4} + \frac{1}{2}n$ und $m = -\frac{1}{4}$ von (160) liefert in entsprechender Weise (124).

Mathematics. — *Eine Verallgemeinerung eines Theorems aus der Theorie der* PFAFF'*schen Gleichungen für den einfachsten Fall* $m=2$. I. Von W. VAN DER KULK. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of March 29, 1941.)

Einleitung:

q linear unabhängige PFAFF'sche Gleichungen

$$C^x_\lambda \, d\xi^\lambda = 0; \lambda = 1, \ldots, n; x = \mathrm{p}+1, \ldots \mathrm{n}; p = n-q, \ldots \quad (1)$$

wo die C^x_λ in einer $\mathfrak{U}(\overset{}{\underset{0}{\xi^\varkappa}})$ [1]) analytische Funktionen der $\xi^\varkappa; \varkappa = 1, \ldots, n$ sind, bestimmen in jedem Punkte einer n-dimensionalen Mannigfaltigkeit X_n mit den Koordinaten $\xi^\varkappa$ ein System von ∞^p Linienelementen $d\xi^\varkappa$, eine s. g. *p-Richtung*. Eine m-dimensionale Mannigfaltigkeit X_m in der X_n heisst eine *Integral-X_m* von (1), wenn in jedem ihrer Punkte die tangierende m-Richtung in der p-Richtung von (1) enthalten ist, d. h. wenn die Gleichung

$$C^x_{\lambda_1} \, v^{\lambda_1 \ldots \lambda_m} = 0; \lambda_1, \ldots, \lambda_m = 1, \ldots, n; x = \mathrm{p}+1, \ldots, \mathrm{n} \ . \ . \quad (2)$$

gilt, wo $\llcorner v^{\varkappa_1 \ldots \varkappa_m} \lrcorner$ der einfache Pseudo-m-Vektor der tangierenden m-Richtung der X_m in $\xi^\varkappa$ ist [2]). Man beweist leicht, dass dieser Pseudo-m-Vektor nun auch die Gleichung

$$C^x_{\lambda_1 \lambda_2} \, v^{\lambda_1 \ldots \lambda_m} = 0; C^x_{\lambda_1 \lambda_2} = 2 \, \partial_{[\lambda_1} C^x_{\lambda_2]}; \partial_\mu = \frac{\partial}{\partial \xi^\mu}; x = \mathrm{p}+1, \ldots, \mathrm{n} \quad (3)$$

erfüllt. In den CARTAN'schen Untersuchungen nach der Existenz solcher Integral-X_m gibt es nun ein fundamentales Theorem, das sich für den einfachsten Fall $m=2$ folgendermassen formulieren lässt [3]):

---

[1]) $\mathfrak{U}$ bedeutet stets Umgebung, und $\mathfrak{U}$ (....) Umgebung des eingeklammerten Wertsystems.

[2]) Ein Pseudo-m-Vektor ist ein bis auf einen skalaren Faktor bestimmter m-Vektor. Vgl. J. A. SCHOUTEN und D. J. STRUIK; Einf. in die neueren Methoden der Differentialgeometrie I (weiter zitiert als Einf. I) S. 9 und 15. Das Zeichen $\llcorner \ \lrcorner$ bedeutet „bis auf einen skalaren Faktor"; vgl. D. V. DANTZIG, On the general projective differential-geometry III, Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 150—155, 150 (1934).

[3]) Vgl. z.B. E. CARTAN; Sur l'intégration des systèmes d'équations aux différentielles totales, Ann. Ec. Norm. (3), **18**, 241—311 (1901).

Das System

$$\left.\begin{aligned} & v^{[\mu\lambda} v^{\varrho\sigma]} = 0;\ \mu, \lambda, \varrho, \sigma = 1, \ldots, n,\ ^{4)} \\ & C^x_\mu v^{\mu\lambda} = 0;\ x = p+1, \ldots, n, \\ & C^x_{\mu\lambda} v^{\mu\lambda} = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

habe für jeden Punkt $\xi^\varkappa$ einer $\mathfrak{U}(\overset{}{\underset{0}{\xi^\varkappa}})$ ∞^d Lösungen $\llcorner v^{\mu\lambda} \lrcorner$. In dem projektiven, $(\frac{1}{2}n(n-1)-1)$-dimensionalen Raum $\mathfrak{P}^2$ aller Pseudobivektoren des Punktes $\xi^\varkappa$ bilden diese Lösungen also eine d-dimensionale Punktmenge, die wir $\mathfrak{S}^2_d$ nennen. Die Menge dieser $\mathfrak{S}^2_d$ aller Punkte $\xi^\varkappa$ aus $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ nennen wir ein $\mathfrak{S}^2_d$-*Feld*, und (4) ein System von Gleichungen dieses Feldes. Es mögen ferner in jedem Punkte $\xi^\varkappa$ aus $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ die Pseudovektoren $\llcorner v^\varkappa \lrcorner$, für welche das System (4), zusammen mit

$$v^{[\varkappa} v^{\mu\lambda]} = 0 \quad \ldots \ldots \ldots \quad (5)$$

mindestens eine nichtverschwindende Lösung $\llcorner v^{\mu\lambda} \lrcorner$ besitzt, in dem projektiven, $(n-1)$-dimensionalen Raum $\mathfrak{P}^1$ aller Pseudovektoren des Punktes $\xi^\varkappa$ eine t-dimensionale Punktmenge bilden, die wir $\mathfrak{R}_t$ nennen werden. Die $\mathfrak{R}_t$ wird also gebildet durch alle Richtungen $\llcorner v^\varkappa \lrcorner$, die enthalten sind in den ∞^d Zweirichtungen, die gehören zu den Punkten $\llcorner v^{\mu\lambda} \lrcorner$ der lokalen $\mathfrak{S}^2_d$ in $\xi^\varkappa$. Es gilt nun der fundamentale, von CARTAN herrührende Satz [5])

Fällt in jedem Punkt $\xi^\varkappa$ aus einer $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ die $\mathfrak{R}_t$ zusammen mit der p-Richtung von (1), *so ist das $\mathfrak{S}^2_d$-Feld mit den Gleichungen* (4) *vollständig integrabel; d. h. es gibt zu jedem Wertsystem $\underset{0}{\xi^\varkappa}$, $\underset{0}{v^{\mu\lambda}} \neq 0$, das dem System* (4) *genügt, mindestens eine Integral-X_2, die $\underset{0}{\xi^\varkappa}$ enthält, und in diesem Punkt die Zweirichtung von $\llcorner \underset{0}{v^{\mu\lambda}} \lrcorner$ tangiert.*

Fällt aber $\mathfrak{R}_t$ nicht mit der p-Richtung von (1) zusammen (und dies ist im allgemeinen für $q \geqq \frac{n-1}{2}$ der Fall), so sagt dieses Theorem

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[4]) Diese Gleichung besagt, dass $\llcorner v^{\mu\lambda} \lrcorner$ einfach ist, und somit eine Zweirichtung darstellt. Vgl. Einf. I, S. 18.

[5]) CARTAN ergänzt dieses Fundamentaltheorem durch Hinzufügung von Sätzen, die die Integral-X_2 durch geschickt gewählte Anfangsbedingungen eindeutig festlegen. In weiteren Untersuchungen gibt CARTAN eine Methode um durch Einführung von neuen Variablen jedes PFAFF'sche System auf ein PFAFF'sches System in N Variablen ($N > n$) zurückzuführen, auf welches das Fundamentaltheorem sofort anwendbar ist. Mit diesen Untersuchungen werden wir uns hier aber nicht beschäftigen.

nichts aus über die Existenz der Integral-X_2, obgleich das $\mathfrak{S}_d^2$-Feld wohl Integral-X_2 besitzen kann.

Fast zugleicherzeit und unabhängig von einander haben J. M. THOMAS, E. KÄHLER und C. BURSTIN das CARTAN'sche Theorem in einem zuerst von GOURSAT angegebenen Sinne verallgemeinert. [6]) Dabei ist KÄHLER wohl am gründlichsten vorgegangen. Statt (2) und (3) betrachtet er das System

$$\left.\begin{array}{l} v^{[\varkappa_1\ldots\varkappa_m}\, v^{\mu_1]\ldots\mu_m} = 0 \\ \overset{\alpha_1}{w}_{\lambda_1}\, v^{\lambda_1\ldots\lambda_m} = 0\,;\ \alpha_1 = 1,\ldots, \mathrm{N}_1\,;\ \mathrm{N}_1 = q, \\ \overset{\alpha_2}{w}_{\lambda_1\lambda_2}\, v^{\lambda_1\ldots\lambda_m} = 0\,;\ \alpha_2 = \mathrm{N}_1+1,\ldots, \mathrm{N}_2, \\ \vdots \\ \overset{\alpha_m}{w}_{\lambda_1\ldots\lambda_m}\, v^{\lambda_1\ldots\lambda_m} = 0\,;\ \alpha_m = \mathrm{N}_{m-1}+1,\ldots, \mathrm{N}_m, \end{array}\right\} \quad \ldots \quad (6)$$

und stellt für dieses System von Gleichungen ein analoges Fundamentaltheorem auf. Die Verallgemeinerung besteht also darin, dass er ausser den Gleichungen (2) und (3), die entstehen durch Nullsetzen der Ueberschiebungen von $v^{\varkappa_1\ldots\varkappa_m}$ mit q kovarianten Vektoren und deren Rotationen, auch Gleichungen in Betracht zieht, die entstehen durch Nullsetzen der Ueberschiebungen von $v^{\varkappa_1\ldots\varkappa_m}$ mit beliebigen kovarianten r-Vektoren; $r = 2, 3, \ldots, m$. Eine X_m, deren tangierende m-Richtung $\llcorner v^{\varkappa_1\ldots\varkappa_m} \lrcorner$ in jedem Punkte dem System (6) genügt, nennt er eine *Integral-X_m* dieses Systems. Es lässt sich zeigen, dass $\llcorner v^{\varkappa_1\ldots\varkappa_m} \lrcorner$ dann auch die Gleichungen

$$\left.\begin{array}{r} 2\,(\partial_{[\lambda_1}\, \overset{\alpha_1}{w}_{\lambda_2]})\, v^{\lambda_1\ldots\lambda_m} = 0\,;\ \alpha_1 = 1,\ldots, \mathrm{N}_1\,;\ \mathrm{N}_1 = q \\ \vdots \\ (m-1)\,(\partial_{[\lambda_1}\, \overset{\alpha_{m-1}}{w}_{\lambda_2\ldots\lambda_m]})\, v^{\lambda_1\ldots\lambda_m} = 0\,;\ \alpha_{m-1} = \mathrm{N}_{m-2}+1,\ldots, \mathrm{N}_{m-1} \end{array}\right\} \quad (7)$$

erfüllt, und man kann also diese Gleichungen zu dem System (6) hinzufügen. Wendet man auf das neue System (6) denselben Prozess an,

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[6]) Vgl. E. COURSAT, Sur certaines systèmes d'équations aux différentielles totales et sur une généralisation du problème de PFAFF, Toulouse Ann. (3) **7** 1—58 (1917), und Leçons sur le problème de PFAFF, Paris, J. Hermann, S. 111 (1922), J. M. THOMAS, An existence theorem for generalized PFAFFian systems, und The condition for a PFAFFian system in involution; Bull. Amer. Soc. **40**, 309—320 (1934), E. KÄHLER; Einf. in die Theorie der Systeme von Differentialgleichungen; Hamb. Math. Einzelschr. **16** B. G. Teubner, Leipzig, 1934, C. BURSTIN, Beiträge zum Problem von PFAFF und zur Theorie der PFAFF'schen Aggregate, I Beitrag, Rec. math. Moscou, **41**, 582—618 (1935), J. M. THOMAS, Differential systems, Coll. Publ. Am. Math. Soc., Vol. **21** (1937).

d.h. bildet man für dieses System die Gleichungen (7), und fügt man sie zu diesem System hinzu, so lässt das System sich dadurch nicht mehr verlängern. Ein solches System nennt KÄHLER *vollständig*. Das Fundamentaltheorem von KÄHLER lässt sich nun für den einfachsten Fall $m=2$ folgendermassen formulieren:

Es sei ein $\mathfrak{S}_d^2$-Feld gegeben mit den Gleichungen

$$\left.\begin{array}{l} v^{[\mu\lambda}\, v^{\varrho\sigma]}=0, \\ \overset{\alpha_1}{w}_{\mu}\, v^{\mu\lambda}=0\,;\ \alpha_1=1,\ldots,\mathrm{N}_1, \\ \overset{\alpha_2}{w}_{\mu\lambda}\, v^{\mu\lambda}=0\,;\ \alpha_2=\mathrm{N}_1+1,\ldots,\mathrm{N}_2, \end{array}\right\} \quad \ldots\ldots \quad (8)$$

und es sei (8) ein vollständiges System. Man bilde nun das zu diesem $\mathfrak{S}_d^2$-Felde gehörige $\mathfrak{R}_t$-Feld. Sodann gilt das Theorem:

Fällt das $\mathfrak{R}_t$-Feld mit dem p-Richtungsfeld der $\overset{\alpha_1}{w}_{\mu}$; $\alpha_1=1,\ldots,\mathrm{N}_1$; $\mathrm{N}_1=\mathrm{q}$ zusammen, so ist das $\mathfrak{S}_d^2$-Feld vollständig integrabel [7].

Fällt aber das $\mathfrak{R}_t$-Feld nicht mit diesem p-Richtungsfeld zusammen, so gestattet das Theorem von KÄHLER keine Aussage über die Existenz von Integral-X_2, obgleich es sehr wohl solche Integral-X_2 geben kann, und das $\mathfrak{S}_d^2$-Feld sogar vollständig integrabel sein kann. Diese Ansätze können nun für $m=2$ folgendermassen verallgemeinert werden:

Es sei ein beliebiges $\mathfrak{S}_d^2$-Feld gegeben mit den Gleichungen

$$\left.\begin{array}{l} v^{[\mu\lambda}\, v^{\varrho\sigma]}=0 \\ \overset{i}{F}(\xi^{\varkappa}, v^{\mu\lambda})=0\,;\ i=\mathrm{d}+1,\ldots,2\,(\mathrm{n}-2), \end{array}\right\} \quad \ldots\ldots \quad (9)$$

wo die $\overset{i}{F}$ analytisch sind in einer Umgebung einer Nullstelle $\underset{0}{\xi^{\nu}}, \underset{0}{v^{\mu\lambda}} \neq 0$ von (9), und ausserdem homogen in den $v^{\mu\lambda}$. Eine X_2 heisse eine Integral-X_2 des Systems (9), wenn der Bivektor der tangierenden Zweirichtung in jedem Punkte der X_2 dem System (9) genügt. Es lässt sich nun zeigen, dass für eine solche X_2 die linearen Gleichungen

$$v^{\varrho\omega}\,\{\bar{\partial}_{\omega}\overset{i}{F}+\overset{i}{F}_{\varkappa\mu}\, v^{\lambda\mu}\, Z^{\varkappa}_{\omega\lambda}\}=0\,;\ i=\mathrm{d}+1,\ldots,2\,(\mathrm{n}-2), \quad \ldots \quad (10)$$

mit den in den Indizes ω und λ symmetrischen Unbekannten $Z^{\varkappa}_{\omega\lambda}$, mindestens eine Lösung haben. Dabei bedeutet

$$\bar{\partial}_{\omega}\overset{i}{F}=\frac{\partial \overset{i}{F}}{\partial \xi^{\omega}}\ (v^{\mu\lambda}\ \textit{nicht}\ \text{differenzieren}). \quad \ldots\ldots \quad (11)$$

[7]) Auch KÄHLER ergänzt dieses Theorem durch Aufstellung solcher Anfangsbedingungen, die die Integral-X_2 eindeutig festlegen.

und

$$\overset{i}{F}_{\mu\lambda} = \frac{\partial \overset{i}{F}}{\partial v^{\mu\lambda}};\ i = \mathrm{d}+1, \ldots, 2\,(\mathrm{n}-2) \ldots\ldots \quad (12)$$

Analog mit KÄHLER heisse nun das $\mathfrak{S}^2_d$-Feld *völlstandig*, wenn die Gleichungen (10) für jedes Wertsystem $\xi^\varkappa$, $v^{\mu\lambda} \neq 0$ das den Gleichungen (9) genügt, mindestens eine Lösung haben.

Man bilde das zum $\mathfrak{S}^2_d$-Felde gehörige $\mathfrak{R}_t$-Feld. Die lokale $\mathfrak{R}_t$ in einem beliebigen Punkt $\xi^\varkappa$ ist eine Regelfläche, denn sie wird gebildet durch die ∞^d in der lokalen $\mathfrak{P}^1$ liegenden Geraden, die den Punkten $\lfloor v^{\mu\lambda} \rfloor$ der lokalen $\mathfrak{S}^2_d$ zugeordnet sind. Aendert sich längs jeder dieser Geraden der Tangentialraum der $\mathfrak{R}_t$ nicht, so heisst die $\mathfrak{R}_t$ *abwickelbar*.

In dieser Arbeit soll nun folgendes Theorem bewiesen werden:

Ein vollständiges $\mathfrak{S}^2_d$-Feld, dessen $\mathfrak{R}_t$-Feld abwickelbar ist, ist vollständig integrabel,

und es sollen ferner Anfangsbedingungen aufgestellt werden, welche eine Integral-X_2 eindeutig festlegen.

Ist aber das $\mathfrak{R}_t$-Feld nicht abwickelbar, so sagt das Theorem nichts aus über die Integral-X_2 des Feldes, obgleich das Feld auch dann noch vollständig integrabel sein kann.

Das Theorem ist in zweierlei Hinsicht eine Verallgemeinerung der obenerwähnten Theoreme von CARTAN und KÄHLER. Erstens werden bei CARTAN und KÄHLER a priori nur solche $\mathfrak{S}^2_d$-Felder in Betracht gezogen, die in dem projektiven Raum aller Pseudobivektoren des Punktes $\xi^\varkappa$ vollständiger Durchschnitt sind der $2\,(n-2)$-dimensionalen Mannigfaltigkeit der einfachen Pseudobivektoren

$$v^{[\mu\lambda}\, v^{\varrho\sigma]} = 0 \ldots\ldots\ldots \quad (13)$$

mit einem linearen Raum. Denn in (4) und (8) kommen ausser (13) nur Gleichungen vor, die linear sind in den $v^{\mu\lambda}$. Dagegen brauchen im angekündigten Theorem die Funktionen $\overset{i}{F}$ gar nicht linear in den $v^{\mu\lambda}$ zu sein: sie brauchen sogar nicht rational in den $v^{\mu\lambda}$ zu sein. Zweitens fordern CARTAN und KÄHLER dass die lokale $\mathfrak{R}_t$ in jedem Punkte ein linearer Raum ist, während hier nur Abwickelbarkeit gefordert wird.

Der Anlass zu der vorliegenden Arbeit war ein Schönheitsfehler, der der Theorie von KÄHLER anhaftet. Gibt man nämlich ein $\mathfrak{S}^2_d$-Feld durch zwei gleichwertige Systeme von der Form (8), und wendet man auf jedes dieser Systeme das Theorem von KÄHLER an, so kann es geschehen, dass man zwei ganz verschiedene Resultate erhält. Z.B. betrachte man in einer X_4 das $\mathfrak{S}^2_2$-Feld mit den Gleichungen

$$w_\mu\, v^{\mu\lambda} = 0 \ldots\ldots\ldots \quad (14)$$

Wendet man auf dieses System die Theorie von KÄHLER an, so folgt, dass das System

$$\left.\begin{aligned} w_\mu v^{\mu\lambda} &= 0, \\ 2\,(\partial_{[\mu} w_{\lambda]})\, v^{\mu\lambda} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15)$$

vollständig integrabel ist, womit eine Uebersicht über die Integral-X_2 des $\mathfrak{S}_2^2$-Feldes gewonnen ist. Man kann aber das $\mathfrak{S}_2^2$-Feld auch durch die Gleichungen

$$w_{[\mu} \overset{\varkappa}{e}_{\lambda]}\, v^{\mu\lambda} = 0;\ \varkappa = 1, 2, 3, 4 \quad \ldots\ldots \quad (16)$$

angeben. Zwar ist das System (16) vollständig, da aber das zugehörige $\mathfrak{R}_t$-Feld nicht den ganzen projektiven Raum der Pseudovektoren $\lfloor v^\varkappa \rfloor$ erfüllt, ist das KÄHLER'sche Theorem hier nicht anwendbar. Diese Schwierigkeit [8]) besteht beim angekündigten Theorem nicht, denn die Lösbarkeit des Systems (10) ändert sich nicht, wenn man (9) durch ein gleichwertiges System derselben Art ersetzt.

Es liegt auf der Hand diese Theorie nun auch für $m > 2$ zu entwickeln. Man kann beweisen, dass das Theorem jedenfalls gültig ist für $m = 3$ und $d = 1$ oder 2. Es ist aber nicht wahrscheinlich, dass für andere Werte von m und d die Abwickelbarkeit des $\mathfrak{R}_t$-Feldes allein schon für die vollständige Integrabilität eines vollsständigen $\mathfrak{S}_d^m$-Feldes hinreichend ist.

In dieser ersten Mitteilung werden $\mathfrak{S}_d^2$-Felder und ihre zugehörigen $\mathfrak{R}_t$-Felder definiert und einige Eigenschaften dieser Felder bewiesen.

1. *Das $\mathfrak{S}_d^2$-Feld und das zugehörige $\mathfrak{R}_t$-Feld.*

Wir betrachten $\frac{1}{2}n(n-1) - 1 - d$ Funktionen der n Variablen $\xi^\varkappa$; $\varkappa = 1, \ldots, n$ und der $N = \frac{1}{2}n(n-1)$ Variablen $v^{\mu\lambda}$; $v^{\mu\lambda} = -v^{\lambda\mu}$; $\mu, \lambda = 1, \ldots, n$

$$\overset{k}{F}(\xi^\varkappa, v^{\mu\lambda});\ \varkappa, \mu, \lambda = 1, \ldots, n;\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1, \quad \ldots \quad (1.1)$$

die analytisch sind in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ und ausserdem homogen in den $v^{\mu\lambda}$. Es sei das Wertsystem $\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}} \neq 0$ [9]) eine Nullstelle der $\overset{k}{F}$, und es

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[8]) Diese Schwierigkeit wäre aufgehoben, sobald man eine Methode aufstellen könnte, mit deren Hilfe das System (6) sich so schreiben liesse, dass die Gleichungen des Systems von einander unabhängig wären, und zugleicherzeit die Zahlen $N_1, N_2, \ldots, N_{m-1}$ maximal. Denn dann könnte man stets zuerst das System (6) auf eine solche Form bringen, und sodann das Theorem von KÄHLER auf das System in dieser Form anwenden. Eine solche Methode würde z.B. das System (16) automatisch in die Gleichung (14) überführen. Nun lässt sich eine derartige Methode zwar aufstellen (z.B. mit Hilfe der Eliminationstheorie, vgl. B. L. v. d. WAERDEN; Moderne Algebra II, Leipzig, Teubner, Kap. XI), sie ist aber schon für $m = 2$ hinreichend kompliziert.

[9]) $\underset{0}{v^{\mu\lambda}} \neq 0$ bedeutet "nicht alle N Zahlen $\underset{0}{v^{\mu\lambda}}$; $\mu, \lambda = 1, \ldots, n$ verschwinden".

seien für dieses Wertsystem die Ableitungen

$$\overset{k}{F}_{\mu\lambda} \overset{\text{def}}{=} \frac{\partial \overset{k}{F}}{\partial v^{\mu\lambda}};\ \mu, \lambda = 1, \ldots, n;\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1 \quad . \quad . \quad (1.2)$$

linear unabhängig. (d. h. es existiere keine Relation von der Form $\underset{k}{\sigma} \overset{k}{F}_{\mu\lambda} = 0;\ \mu, \lambda = 1, \ldots, n$ wo nicht alle $\underset{k}{\sigma}$ verschwinden.) Sodann lassen sich in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ $N-1-d$ der Variablen $v^{\mu\lambda}$, die wir mit $v^{\alpha\beta}$ bezeichnen wollen, aus den Gleichungen

$$\overset{k}{F}(\xi^\varkappa, v^{\mu\lambda}) = 0;\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1 \quad . \quad . \quad . \quad . \quad (1.3)$$

lösen als analytische Funktionen der $\xi^\varkappa$ und der übrigen $d+1$ der Variablen $v^{\mu\lambda}$, die mit $v^{\varepsilon\eta}$ bezeichnet werden mögen,

$$v^{\alpha\beta} = f^{\alpha\beta}(\xi^\varkappa, v^{\varepsilon\eta}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

In diesen Gleichungen sind die $\xi^\varkappa$ und $v^{\varepsilon\eta}$ innerhalb einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\varepsilon\eta}})$ beliebig wählbar. Die $f^{\alpha\beta}$ sind in den $v^{\varepsilon\eta}$ homogen vom Grade 1. Betrachtet man die $\xi^\varkappa$ als Koordinaten in einer X_n, und die $v^{\mu\lambda}$ als Bestimmungszahlen eines Pseudobivektors in $\xi^\varkappa$, so bestimmen die Gleichungen (1.4) in jedem Punkte $\xi^\varkappa$ einer $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ eine d-dimensionale Mannigfaltigkeit von Pseudobivektoren. Im projektiven $(N-1)$-dimensionalen Raum $\mathfrak{P}^2$ aller Pseudobivektoren des Punktes $\xi^\varkappa$ bildet diese Mannigfaltigkeit also eine d-dimensionale Punktmenge, die wir eine $\mathfrak{M}_d^2$ nennen werden. Die Menge dieser $\mathfrak{M}_d^2$ in allen Punkten aus $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ nennen wir ein in $\mathfrak{U}(\underset{0}{\xi^\varkappa})$ definiertes $\mathfrak{M}_d^2$-Feld. Das $\mathfrak{M}_d^2$-Feld ist also die Nullstellenmannigfaltigkeit der $\overset{k}{F}$ in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$. Diese $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ lässt sich immer so wählen, dass die $\overset{k}{F}_{\mu\lambda};\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1$ für jedes Wertsystem $\xi^\varkappa, v^{\mu\lambda}$ aus $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ linear unabhängig sind. Genügt ein solches Wertsystem ausserdem den Gleichungen (1.3), so stellen die Gleichungen

$$\overset{k}{F}_{\mu\lambda} V^{\mu\lambda} = 0;\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1 \quad . \quad . \quad . \quad . \quad (1.5)$$

($V^{\mu\lambda}$ sind hier die „laufenden" Koordinaten) in der lokalen $\mathfrak{P}^2$ von $\xi^\varkappa$ den d-dimensionalen Tangentialraum der lokalen $\mathfrak{M}_d^2$ im Punkte $\lfloor v^{\mu\lambda} \rfloor$ dar. Die Funktionen $\overset{k}{F};\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1$, nennen wir eine *Basis* des

$\mathfrak{M}^2_d$-Feldes in $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$. Ist $F(\xi^\varkappa, v^{\mu\lambda})$ eine in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ analytische, und in den $v^{\mu\lambda}$ homogene Funktion, die Null ist auf dem $\mathfrak{M}^2_d$-Felde, d. h. die für jede Nullstelle der $\overset{k}{F}$ verschwindet, so gilt infolge eines bekannten Satzes [10]) eine Gleichung von der Form

$$F(\xi^\varkappa, v^{\mu\lambda}) = \underset{k}{\chi}(\xi^\varkappa, v^{\mu\lambda}) \overset{k}{F}(\xi^\varkappa, v^{\mu\lambda}), \quad \ldots \ldots \quad (1.6)$$

mit in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ analytischen Funktionen $\underset{k}{\chi}$. Bilden die Funktionen $\overset{k'}{F}(\xi^\varkappa, v^{\mu\lambda})$; $k' = (\mathrm{d}+1)', \ldots, (\mathrm{N}-1)'$ [11]) eine andere Basis des $\mathfrak{M}^2_d$-Feldes in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$, d. h. also verschwinden sie auf dem Felde, und sind ihre Ableitungen $\overset{k'}{F}_{\mu\lambda}$; $k' = (\mathrm{d}+1)', \ldots, (\mathrm{N}-1)'$, in $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ linear unabhängig, so gelten in einer $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ infolge (1.6) die Gleichungen

$$\left.\begin{aligned}\overset{k'}{F}(\xi^\varkappa, v^{\mu\lambda}) = \overset{k'}{\underset{k}{\chi}}(\xi^\varkappa, v^{\mu\lambda}) \overset{k}{F}(\xi^\varkappa, v^{\mu\lambda});\ k = \mathrm{d}+1, \ldots, \mathrm{N}-1;\\ k' = (\mathrm{d}+1)', \ldots, (\mathrm{N}-1)'\end{aligned}\right\} \quad (1.7)$$

mit nicht verschwindender Determinante der $\overset{k'}{\underset{k}{\chi}}$. Die Gleichungen $\overset{k'}{F} = 0$; $k' = (\mathrm{d}+1)', \ldots, (\mathrm{N}-1)'$ stellen in dieser $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$ also tatsächlich dasselbe $\mathfrak{M}^2_d$-Feld dar als (1.3).

Ein Pseudobivektor $\lfloor v^{\mu\lambda} \rfloor \neq 0$ des Punktes $\xi^\varkappa$ heisst *einfach*, wenn es eine Gleichung von der Form

$$v^{\mu\lambda} = B^{\mu}_{[2}\, B^{\lambda}_{1]} \quad \ldots \ldots \ldots \quad (1.8)$$

gibt. Geometrisch stellt ein einfacher Pseudobivektor eine Zweirichtung in $\xi^\varkappa$ dar. Die n.u.h. Bedingung dafür, dass $v^{\mu\lambda}$ einfach ist, lautet

$$v^{[\mu\lambda}\, v^{\varrho\sigma]} = 0. \quad \ldots \ldots \ldots \quad (1.9)$$

Man beweist leicht, dass sich aus (1.9) in einer $\mathfrak{U}$ einer beliebigen nichtverschwindenden Nullstelle von (1.9) $N-1-2(n-2)$ der Bestimmungszahlen von $v^{\mu\lambda}$ lösen lassen als analytische Funktionen der $2(n-2)+1$ übrigen, sodass in jedem Punkte $\xi^\varkappa$ die einfachen Pseudobivektoren in der lokalen $\mathfrak{P}^2$ eine $\mathfrak{M}^2_{2(n-2)}$ bilden, die wir $\mathfrak{S}^2$ nennen werden. Es lässt

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10) Vgl. E. KÄHLER l.c., S. 12.

11) Anhängen eines Akzents an k, $\mathrm{d}+1$, ..., $\mathrm{N}-1$ bedeutet Aenderung der Indexart, und *nicht* eine Aenderung der Zahlen d und N. (Vgl. Einf. I, S. 1.)

sich ferner zeigen, dass es unter den linken Gliedern von (1.9) $N-1-2(n-2)$ gibt, die wir mit $\overset{r}{\Phi}$; $r=2(n-2)+1, \ldots, N-1$, bezeichnen wollen, deren Ableitungen nach $v^{\mu\lambda}$ in $\underset{0}{\xi^{\varkappa}}$; $\underset{0}{v^{\mu\lambda}} \neq 0$ (wo jetzt $\underset{0}{v^{\mu\lambda}}$ einfach ist.[12])) und somit auch in einer $\mathfrak{U}(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}})$ linear unabhängig sind. Die $\overset{r}{\Phi}$ bilden also in $\mathfrak{U}(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}})$ eine Basis des von den $\mathfrak{S}^2$ gebildeten s.g. $\mathfrak{S}^2$-Feldes.

Ein $\mathfrak{M}_d^2$-Feld, dessen $\mathfrak{M}_d^2$ in jedem Punkte $\xi^{\varkappa}$ in der lokalen $\mathfrak{S}^2$ liegt, und also nur einfache Bivektoren enthält, nennen wir ein $\mathfrak{S}_d^2$-*Feld*. Es ist stets $d \leqq 2(n-2)$. Das $\mathfrak{S}^2$-Feld ist selbst ein $\mathfrak{S}^2_{2(n-2)}$-Feld. Es sei jetzt der einfache Pseudobivektor $\llcorner \underset{0}{v^{\mu\lambda}} \lrcorner \neq 0$ so gewählt, dass $\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}}$ ein Wertsystem eines $\mathfrak{S}_d^2$-Feldes ist, d.h. dass $\llcorner \underset{0}{v^{\mu\lambda}} \lrcorner$ ein Punkt der lokalen $\mathfrak{S}_d^2$ des Feldes in $\underset{0}{\xi^{\varkappa}}$ ist. Sodann lassen sich $2(n-2)-d$ auf dem Felde verschwindende Funktionen $\overset{i}{F}(\xi^{\varkappa}, v^{\mu\lambda})$; $i=d+1, \ldots, 2(n-2)$ konstruieren, die zusammen mit den $\overset{r}{\Phi}$; $r=2(n-2)+1, \ldots, N-1$ in einer $\mathfrak{U}(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}})$ eine Basis des $\mathfrak{S}_d^2$-Feldes bilden. Dazu ist nämlich n.u.h., diese auf dem Felde verschwindenden $\overset{i}{F}$ so zu wählen, dass ihre Ableitungen

$$\overset{i}{F}_{\mu\lambda};\ i=d+1, \ldots, 2(n-2)$$

zusammen mit den $\overset{r}{\Phi}_{\mu\lambda}$; $r=2(n-2)+1, \ldots, N-1$ in $\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}}$ (und folglich auch in einer $\mathfrak{U}(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}})$) linear unabhängig sind, oder anders gesagt, dass die Gleichungen

$$\left.\begin{aligned} &\text{a.}\ (\overset{i}{F}_{\mu\lambda})_0\, V^{\mu\lambda}=0;\ i=d+1, \ldots, 2(n-2), \\ &\text{b.}\ (\overset{r}{\Phi}_{\mu\lambda})_0\, V^{\mu\lambda}=0;\ r=2(n-2)+1, \ldots, N-1;\ (\ )_0=(\ )_{\xi^{\varkappa}=\underset{0}{\xi^{\varkappa}};\ v^{\mu\lambda}=\underset{0}{v^{\mu\lambda}}} \end{aligned}\right\} \quad (1.10)$$

in der lokalen $\mathfrak{P}^2$ von $\underset{0}{\xi^{\varkappa}}$ einen d-dimensionalen Raum darstellen. (Der Tangentialraum der lokalen $\mathfrak{S}_d^2$ im Punkte $\llcorner \underset{0}{v^{\mu\lambda}} \lrcorner$). Man sieht leicht ein, dass eine solche Wahl stets möglich ist. Ein solches System von Funk-

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[12]) Uebrigens darf man das Wertsystem $\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}}$ beliebig wählen. Dieses Wertsystem hat also nichts zu tun mit dem früher erwähnten Wertsystem $\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}}$, das eine Nullstelle der $\overset{k}{F}$; $k=d+1, \ldots, N-1$ war.

tionen $\overset{i}{F}; i = d + 1, \ldots, 2(n-2)$ nennen wir eine $\mathfrak{S}$-*Basis* des Feldes in $\mathfrak{U}(\underset{0}{\xi^\varkappa}, \underset{0}{v^{\mu\lambda}})$. Das $\mathfrak{S}_d^2$-Feld ist in dieser $\mathfrak{U}$ die Nullstellenmannigfaltigkeit der $v^{[\mu\lambda} v^{\varrho\sigma]}$ und der $\overset{i}{F}$. Da die $\overset{r}{\Phi}$ eine Basis des $\mathfrak{S}^2$-Feldes bilden, und sie anderseits in den linken Gliedern von (1.9) enthalten sind, sind die Ableitungen dieser linken Glieder nach $v^{\mu\lambda}$ linear abhängig von den $\overset{r}{\Phi}_{\mu\lambda}$ und umgekehrt. Daraus folgt dass (1.10) und

$$\left.\begin{aligned} &\text{a.}\ (\overset{i}{F}_{\mu\lambda})_0\, V^{\mu\lambda} = 0; \quad i = d+1, \ldots, 2(n-2), \\ &b.\ \underset{0}{v^{[\mu\lambda}}\, V^{\varrho\sigma]} = 0 \end{aligned}\right\} \quad . \quad . \quad (1.11)$$

gleichwertig sind. Nun kann aber $\underset{0}{v^{\mu\lambda}}$ als einfacher Bivektor in der Form

$$\underset{0}{v^{\mu\lambda}} = \underset{0}{B^{\mu}_{[2}}\, \underset{0}{B^{\lambda}_{1]}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.12)$$

geschrieben werden, und die Lösungen von (1.11 *b*) haben somit die Form

$$V^{\mu\lambda} = \underset{0}{B^{[\mu}_{[2}}\, Z^{\lambda]}_{1]}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.13)$$

wo die $Z^\varkappa_a; \varkappa = 1, \ldots, n; a = 1, 2$ beliebig sind. Substitution in (1.11 *a*) ergibt

$$(\overset{i}{F}_{\mu\lambda})_0\, \underset{0}{B^{\mu}_{[2}}\, Z^{\lambda}_{1]} = 0; \quad i = d+1, \ldots, 2(n-2) \quad . \quad . \quad . \quad (1.14)$$

Der durch (1.10) dargestellte lineare Raum besteht somit aus allen Bivektoren von der Form (1.13), wo die $Z^\varkappa_a$ die Gleichungen (1.14) erfüllen. Das System (1.14) hat ersichtlich folgende vier unabhängige Lösungen

$$\left.\begin{aligned} Z^\varkappa_1 &= B^\varkappa_1 \;;\; Z^\varkappa_2 = 0, \\ Z^\varkappa_1 &= B^\varkappa_2 \;;\; Z^\varkappa_2 = 0, \\ Z^\varkappa_1 &= 0 \;;\; Z^\varkappa_2 = B^\varkappa_1, \\ Z^\varkappa_1 &= 0 \;;\; Z^\varkappa_2 = B^\varkappa_2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.15)$$

von denen zwei die triviale Lösung $V^{\mu\lambda} = 0$, und die beiden anderen die Lösung $V^{\mu\lambda} = \underset{0}{v^{\mu\lambda}} \neq 0$ des Systems (1.10) liefern. Da aber der durch (1.10) dargestellte Raum *d*-dimensional ist, hat (1.10) *d* und nicht mehr als *d* Lösungen, die untereinander und von $\underset{0}{v^{\mu\lambda}}$ linear unabhängig sind, oder anders gesagt, es gibt *d* und nicht mehr als *d* Bivektoren von der Form (1.13), die (1.14) genügen, und die mit $\underset{0}{v^{\mu\lambda}}$ ein System von $d+1$ linear unabhängigen Bivektoren bilden. Daraus folgt aber, wie man leicht beweist, dass es *d* und nicht mehr als *d* Lösungen $Z^\varkappa_a$ von (1.14) gibt, die

untereinander und von den vier Lösungen (1. 15) linear unabhängig sind, m.a.W. das System (1. 14), das $2n-4-d$ Gleichungen mit den $2n$ Unbekannten $Z_a^{\varkappa}$ enthält, hat $d+4$ und nicht mehr als $d+4$ linear unabhängige Lösungen. Daraus geht aber hervor, dass die linken Glieder von (1. 14) linear unabhängig sind, wenn man dabei die $Z_a^{\varkappa}$ als Variablen betrachtet. Wie man leicht einsieht, sind die linken Glieder von (1. 14) dann und nur dann linear unabhängig, wenn die linearen Formen

$$(\overset{i}{F}_{\varkappa\mu}\, v^{\lambda\mu})_0\, Z_\lambda^{\varkappa} \quad ; \quad i = d+1, \ldots, 2(n-2), \quad . \quad . \quad . \quad (1.\,16)$$

mit den n^2 Variablen $Z_\lambda^{\varkappa}; \varkappa, \lambda = 1, \ldots, n$ linear unabhängig sind. Der durch (1. 10) dargestellte Raum besteht aus allen Bivektoren von der Form $\underset{0}{v}^{[\mu|\omega|}\, Z_\omega^{\lambda]}$, wo die $Z_\lambda^{\varkappa}$ die linearen Formen (1. 16) annullieren. Damit ist nun folgender Satz bewiesen:

Die auf einem $\mathfrak{S}_d^2$-Felde verschwindenden Funktionen

$$\overset{i}{F}(\xi^{\varkappa}, v^{\mu\lambda}); \quad i = d+1, \ldots, 2(n-2),$$

bilden dann und nur dann in einer Umgebung eines dem $\mathfrak{S}_d^2$-Felde zugehörigen Wertsystems $\underset{0}{\xi}^{\varkappa}, \underset{0}{v}^{\mu\lambda} = \underset{0}{B}{}_{[2}^{\mu}\, \underset{0}{B}{}_{1]}^{\lambda} \neq 0$ eine $\mathfrak{S}$-Basis, wenn die linearen Formen

$$(\overset{i}{F}_{\mu\lambda})_0\, \underset{0}{B}{}_{[2}^{\mu}\, Z_{1]}^{\lambda} \quad ; \quad i = d+1, \ldots, 2(n-2), \quad . \quad . \quad . \quad (1.\,17)$$

in den $2n$ Variablen $Z_a^{\varkappa}; \varkappa = 1, \ldots, n; a = 1, 2$ oder auch die linearen Formen

$$(\overset{i}{F}_{\varkappa\mu}\, v^{\lambda\mu})_0\, Z_\lambda^{\varkappa} \quad ; \quad i = d+1, \ldots, 2(n-2), \quad . \quad . \quad . \quad (1.\,18)$$

in den n^2 Variablen $Z_\lambda^{\varkappa}; \varkappa, \lambda = 1, \ldots, n$, linear unabhängig sind. Der Tangentialraum der lokalen $\mathfrak{S}_d^2$ von $\underset{0}{\xi}^{\varkappa}$ im Punkte $\lfloor \underset{0}{v}^{\mu\lambda} \rfloor$ wird gebildet durch alle Pseudobivektoren von der Form $\underset{0}{B}{}_{[2}^{[\mu}\, Z_{1]}^{\lambda]}$ wo $Z_a^{\varkappa}$ die Formen (1. 17) annulliert, oder auch durch alle Pseudobivektoren von der Form $\underset{0}{v}^{[\mu|\omega|}\, Z_\omega^{\lambda]}$, wo die $Z_\lambda^{\varkappa}$ die Formen (1. 18) annullieren.

Die Pseudovektoren $\lfloor v^{\varkappa} \rfloor$ des Punktes $\xi^{\varkappa}$ bilden einen projektiven, $(n-1)$-dimensionalen Raum, die lokale $\mathfrak{P}^1$ des Punktes $\xi^{\varkappa}$. Ein einfacher Pseudobivektor in $\xi^{\varkappa}$

$$\lfloor v^{\mu\lambda} \rfloor = \lfloor B_{[2}^{\mu}\, B_{1]}^{\lambda} \rfloor \neq 0 \ ; \quad \mu, \lambda = 1, \ldots, n \quad . \quad . \quad . \quad (1.\,19)$$

lässt sich in dieser $\mathfrak{P}^1$ geometrisch als eine Gerade deuten, nämlich als

die Gerade mit den Parametergleichungen

$$\lfloor v^{\varkappa} \rfloor = \lfloor \sigma^a B_a^{\varkappa} \rfloor \;;\quad \varkappa = 1, \ldots, n;\; a = 1, 2, \quad . \quad . \quad . \quad (1.20)$$

wo die σ^a beliebig wählbar sind. Jedem der ∞^d Punkte $\lfloor v^{\mu\lambda} \rfloor$ einer $\mathfrak{S}_d^2$ in der $\mathfrak{P}^2$ von $\xi^{\varkappa}$ ist also eine Gerade in $\mathfrak{P}^1$ zugeordnet. Die Punkte dieser ∞^d Geraden bilden eine Regelfläche, die wir $\mathfrak{R}_t$ nennen werden (t ist die Dimension). Einem $\mathfrak{S}_d^2$-Feld ist also ein $\mathfrak{R}_t$-Feld zugeordnet. Wir können die $\mathfrak{U}\,(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}})$ wo wir das $\mathfrak{S}_d^2$-Feld betrachten, so wählen, dass in allen in Betracht kommenden Punkten $\xi^{\varkappa}$ die zugehörigen $\mathfrak{R}_t$ stets dieselbe Dimension, nämlich t, haben, und ausserdem können wir die Umgebung, in welcher wir die Variablen $\xi^{\varkappa}$ und v^{μ} betrachten, so wählen, dass in dieser Umgebung die $\mathfrak{R}_t$ von jedem Punkte $\xi^{\varkappa}$ keine Singularitäten enthält, und also in jedem ihrer Punkte einen t-dimensionalen Tangentialraum besitzt. Es lässt sich nun folgender Satz beweisen:

Ist $\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}} = \underset{0}{B}{}^{\mu}_{[2} \underset{0}{B}{}^{\lambda}_{1]} \neq 0$ *ein beliebiges Wertsystem eines* $\mathfrak{S}_d^2$*-Feldes, und ist* $\lfloor \underset{0}{v^{\varkappa}} \rfloor = \lfloor \underset{0}{\sigma^a} \underset{0}{B}{}_a^{\varkappa} \rfloor \neq 0$ *ein beliebiger Punkt auf der dem Bivektor* $\underset{0}{v^{\mu\lambda}}$ *zugeordneten Gerade, so wird in der lokalen* $\mathfrak{P}^1$ *von* $\underset{0}{\xi^{\varkappa}}$ *der Tangentialraum der lokalen* $\mathfrak{R}_t$ *im Punkte* $\lfloor \underset{0}{v^{\varkappa}} \rfloor$ *gebildet durch alle Pseudovektoren von der Form* $\lfloor \underset{0}{\sigma^a} Z_a^{\varkappa} \rfloor$, *wo die* $Z_a^{\varkappa}$ *die Formen* (1.17) *annullieren, oder auch durch alle Vektoren von der Form* $\lfloor \underset{0}{v^{\lambda}} Z_{\lambda}^{\varkappa} \rfloor$, *wo die* $Z_{\lambda}^{\varkappa}$ *die Formen* (1.18) *annullieren.*

Dieser Satz, der sich mit Hilfe einer Parameterdarstellung des $\mathfrak{S}_d^2$-Feldes beweisen lässt, gibt eine Methode um tatsächlich die Zahl t zu berechnen, und eine $\mathfrak{U}$ für die Variablen $\xi^{\varkappa}$, $v^{\mu\lambda}$, v^{ϱ} zu bestimmen, in welcher jede $\mathfrak{R}_t$ t-dimensional ist, und keine Singularitäten enthält. Jedem Wertsystem $\xi^{\varkappa}$, $v^{\mu\lambda} \neq 0$, $v^{\varrho} \neq 0$ wo $\xi^{\varkappa}$, $v^{\mu\lambda}$ ein Wertsystem des Feldes ist, und $\lfloor v^{\varkappa} \rfloor$ ein Punkt auf der Gerade von $\lfloor v^{\mu\lambda} \rfloor$, lässt sich nämlich eine Zahl τ zuordnen: die Dimension des linearen Raumes der Punkte $\lfloor v^{\lambda} Z_{\lambda}^{\varkappa} \rfloor$, wo $Z_{\lambda}^{\varkappa}$ den Gleichungen $\overset{i}{F}_{\varkappa\mu}\, v^{\lambda\mu} Z_{\lambda}^{\varkappa} = 0$ genügt. t ist der maximale Wert von τ. Ist $\underset{0}{\xi^{\varkappa}}$, $\underset{0}{v^{\mu\lambda}}$, $\underset{0}{v^{\varrho}}$ ein solches Wertsystem, dessen τ den maximalen Wert t hat, so gibt es eine $\mathfrak{U}\,(\underset{0}{\xi^{\varkappa}}, \underset{0}{v^{\mu\lambda}}, \underset{0}{v^{\varrho}})$ mit den gewünschten Eigenschaften.

Für die Zahlen n, d und t gilt folgende Ungleichung

$$1 + \frac{d}{2} \leqq t \leqq min\,(d+1, n-1) \quad . \quad . \quad . \quad . \quad . \quad (1.21)$$

Die Betrachtungen dieses Paragraphen lassen sich auch für m-Vektoren, mit $m > 2$, aufstellen.

Mathematics. — *Sur des séries et des intégrales définies contenantes les fonctions de* BESSEL. I. By J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 29, 1941.)

Nous avons déjà trouvé la formule suivante [1]):

$$\frac{I_{\nu+\varrho+1}(x)}{\varrho+1}=\int_0^x I_\nu(x-\alpha)\,I_{\varrho+1}(\alpha)\,\frac{d\alpha}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

ν et ϱ étant des nombres arbitraires, dont la partie réelle est plus grande que -1, ce que nous écrirons ainsi: $R(\nu)>-1$ et $R(\varrho)>-1$.

Maintenant nous montrerons qu'on peut faire plusieurs applications de cette formule, qui conduisent aux résultats très importants.

§ 1. D'après les séries absolument convergentes

$$2\sum_{n=0}^{\infty}(-1)^n I_{2n+1}(x)=\sin x,\quad \sum_{n=0}^{\infty}\varepsilon_{2n}(-1)^n I_{2n}(x)=\cos x,\quad \sum_{n=0}^{\infty}\varepsilon_{2n} I_{2n}(x)=1$$

(où $\varepsilon_{2n}=2$ pour $n>0$ et $\varepsilon_0=1$), et les suivantes qu'on peut déduire des dernières: $\sum_{n=0}^{\infty}\varepsilon_{2n} I_{4n}(x)=\cos^2\frac{1}{2}x$ et $2\sum_{n=0}^{\infty} I_{4n+2}(x)=\sin^2\frac{1}{2}x$, nous posons d'abord dans (a) $\nu=2n+1$; après la multiplication des deux membres par $2(-1)^n(\varrho+1)$ et la sommation sur n de 0 à ∞ nous trouvons:

$$2\sum_{n=0}^{\infty}(-1)^n I_{\varrho+2n+2}(x)=(\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\sin(x-\alpha)\frac{d\alpha}{\alpha},\quad R(\varrho)>-1. \quad . \quad (1)$$

Alors nous substituons dans (a) $\nu=2n$ et multiplions les deux membres par $\varepsilon_{2n}(-1)^n(\varrho+1)$ resp. $\varepsilon_{2n}(\varrho+1)$; après la sommation sur n de 0 à ∞ on trouve les formules suivantes:

$$\sum_{n=0}^{\infty}\varepsilon_{2n}(-1)^n I_{\varrho+2n+1}(x)=(\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha},\quad R(\varrho)>-1 \quad . \quad (2)$$

$$\sum_{n=0}^{\infty}\varepsilon_{2n} I_{\varrho+2n+1}(x)=(\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\frac{d\alpha}{\alpha},\quad R(\varrho)>-1 \quad . \quad . \quad (3)$$

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[1]) Nieuw Archief voor Wiskunde (2) VII, 1907, p. 404, (38).

Ensuite nous substituons dans (a) $\nu = 4n$ resp. $\nu = 4n+2$ et multiplions les deux membres par $\varepsilon_{2n}(\varrho+1)$ resp. $2(\varrho+1)$; après la sommation sur n de 0 à ∞ on trouve:

$$\sum_{n=0}^{\infty} \varepsilon_{2n} I_{\varrho+4n+1}(x) = (\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\cos^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha},\ R(\varrho) > -1\ . \quad . \quad (4)$$

$$2\sum_{n=0}^{\infty} I_{\varrho+4n+3}(x) = (\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\sin 2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha},\ R(\varrho) > -1 \quad (5)$$

Après la multiplication des deux membres de (2) par $\sin x$ resp. $\cos x$ et ceux de (1) par $\cos x$ resp. $\sin x$, nous trouvons après la soustraction resp. l'addition des membres correspondants:

$$\left.\begin{aligned} &2\sum_{n=0}^{\infty}(-1)^n\{\sin x\, I_{\varrho+2n+1}(x) - \cos x\, I_{\varrho+2n+2}(x)\} - \\ &- \sin x\, I_{\varrho+1}(x) = (\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\sin\alpha\frac{d\alpha}{\alpha},\ R(\varrho) > -1 \end{aligned}\right\} \quad . \quad . \quad (6)$$

$$\left.\begin{aligned} &2\sum_{n=0}^{\infty}(-1)^n\{\cos x\, I_{\varrho+2n+1}(x) + \sin x\, I_{\varrho+2n+2}(x)\} - \\ &- \cos x\, I_{\varrho+1}(x) = (\varrho+1)\int_0^x I_{\varrho+1}(\alpha)\cos\alpha\frac{d\alpha}{\alpha},\ R(\varrho) > -1. \end{aligned}\right\} \quad . \quad . \quad (7)$$

Des séries indiquées aux commencement on peut déduire, pour m entier positif ou nulle, les suivantes:

$$2\sum_{n=0}^{\infty}(-1)^n I_{2m+2n+1}(x) = (-1)^m\{\sin x - 2\sum_{n=0}^{m-1}(-1)^n I_{2n+1}(x)\}, \quad . \quad (b)$$

$$2\sum_{n=0}^{\infty}(-1)^n I_{2m+2n}(x) = (-1)^m\{\cos x + I_0(x) - 2\sum_{n=0}^{m-1}(-1)^n I_{2n}(x)\}, \quad (c)$$

$$2\sum_{n=0}^{\infty} I_{2m+2n}(x) = 1 + I_0(x) - 2\sum_{n=0}^{m-1} I_{2n}(x), \quad . \quad . \quad . \quad . \quad (d)$$

$$\sum_{n=0}^{\infty}\varepsilon_{2n} I_{4m+4n}(x) = \cos^2\tfrac{1}{2}x + I_0(x) - 2\sum_{n=0}^{m-1} I_{4n}(x), \quad . \quad . \quad . \quad (e)$$

$$2\sum_{n=0}^{\infty} I_{4m+4n+2}(x) = \sin^2\tfrac{1}{2}x - 2\sum_{n=0}^{m-1} I_{4n+2}(x). \quad . \quad . \quad . \quad . \quad (f)$$

Alors nous trouvons en substituant dans (1) $\varrho = 2m-1$ resp. $\varrho = 2m-2$, dans (2) $\varrho = 2m$ resp. $\varrho = 2m-1$, dans (3) $\varrho = 2m-1$, dans (4) $\varrho = 4m-1$ resp. $\varrho = 4m+1$, dans (5) $\varrho = 4m-3$ resp. $\varrho = 4m+1$:

$$\int_0^x I_{2m}(\alpha)\sin(x-\alpha)\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m}\{\sin x-2\sum_{n=0}^{m-1}(-1)^n I_{2n+1}(x)\},\ m>0 \quad (8)$$

$$\int_0^x I_{2m-1}(\alpha)\sin(x-\alpha)\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m-1}\{\cos x+I_0(x)-2\sum_{n=0}^{m-1}(-1)^n I_{2n}(x)\},\ m>0 \quad (9)$$

$$\int_0^x I_{2m+1}(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m+1}\{\sin x-2\sum_{n=0}^{m-1}(-1)^n I_{2n+1}(x)\}-\frac{I_{2m+1}(x)}{2m+1},\ m\geqq 0 \quad (10)$$

$$\int_0^x I_{2m}(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m}\{\cos x+I_0(x)-2\sum_{n=0}^{m-1}(-1)^n I_{2n}(x)\}-\frac{I_{2m}(x)}{2m},\ m>0 \quad (11)$$

$$\int_0^x I_{2m}(\alpha)\frac{d\alpha}{\alpha}=\frac{1}{2m}\{1+I_0(x)-2\sum_{n=0}^{m-1}I_{2n}(x)-I_{2m}(x)\},\ m>0. \quad (12)$$

$$\int_0^x I_{4m}(\alpha)\cos^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha}=\frac{1}{4m}\{\cos^2\tfrac{1}{2}x+I_0(x)-2\sum_{n=0}^{m-1}I_{4n}(x)-I_{4m}(x)\},\ m>0 \quad (13)$$

$$\int_0^x I_{4m+2}(\alpha)\cos^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha}=\frac{1}{4m+2}\{\sin^2\tfrac{1}{2}x-2\sum_{n=0}^{m-1}I_{4n+2}(x)-I_{4m+2}(x)\},\ m\geqq 0 \quad (14)$$

$$\int_0^x I_{4m-2}(\alpha)\sin^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha}=\frac{1}{4m-2}\{\cos^2\tfrac{1}{2}x+I_0(x)-2\sum_{n=0}^{m-1}I_{4n}(x)\},\quad m>0 \quad (15)$$

$$\int_0^x I_{4m}(\alpha)\sin^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha}=\frac{1}{4m}\{\sin^2\tfrac{1}{2}x-2\sum_{n=0}^{m-1}I_{4n+2}(x)\},\ m>0 \quad . \quad (16)$$

En remplaçant dans (10) et (14) m par $m-1$, on obtient:

$$\int_0^x I_{2m-1}(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m-1}\{2\sum_{n=0}^{m-2}(-1)^n I_{2n+1}(x)-\sin x\}-\frac{I_{2m-1}(x)}{2m-1},\ m>0 \quad (10')$$

$$\int_0^x I_{4m-2}(\alpha)\cos^2\tfrac{1}{2}(x-\alpha)\frac{d\alpha}{\alpha}=\frac{1}{4m-2}\{\sin^2\tfrac{1}{2}x-2\sum_{n=0}^{m-2}I_{4n+2}(x)-I_{4m-2}(x)\},\ m>0. \quad (14')$$

Après la multiplication des deux membres de (10′) par sin x resp. cos x et ceux de (9) par cos x resp. sin x on trouve par soustraction resp. addition des membres correspondants:

$$\ldots_{-1}(\alpha)\sin\alpha\,\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m-1}\left[\cos x\,I_0(x)-1+2\sum_{n=0}^{m-2}(-1)^n\{\sin x\,I_{2n+1}(x)-\cos x\,I_{2n+2}(x)\}\right]-\frac{\sin x\,I_{2m-1}(x)}{2m-1},\quad m>0 \qquad (17)$$

$$\ldots_{-1}(\alpha)\cos\alpha\,\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m-1}\left[2\sum_{n=0}^{m-2}(-1)^n\{\cos x\,I_{2n+1}(x)+\sin x\,I_{2n+2}(x)\}-\sin x\,I_0(x)\right]-\frac{\cos x\,I_{2m-1}(x)}{2m-1},\quad m>0 \qquad (18)$$

En faisant de même à l'égard de (11) et (8) on obtient:

$$\int_0^x I_{2m}(\alpha)\sin\alpha\,\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m}\left[\sin x\,I_0(x)-2\sum_{n=0}^{m-1}\{\sin x\,I_{2n}(x)-\cos x\,I_{2n+1}(x)\}\right]-\frac{\sin x\,I_{2m}(x)}{2m},\quad m>0 \qquad (19)$$

$$\int_0^x I_{2m}(\alpha)\cos\alpha\,\frac{d\alpha}{\alpha}=\frac{(-1)^m}{2m}\left[1+\cos x\,I_0(x)-2\sum_{n=0}^{m-1}\{\cos x\,I_{2n}(x)+\sin x\,I_{2n+1}(x)\}\right]-\frac{\cos x\,I_{2m}(x)}{2m},\quad m>0 \qquad (20)$$

En substituant dans (8) jusqu'à (20) $m=1$ on trouve les formules particulières:

$$\int_0^x I_2(\alpha)\sin(x-\alpha)\frac{d\alpha}{\alpha}=I_1(x)-\tfrac{1}{2}\sin x\,, \quad \ldots\ldots \quad (21)$$

$$\int_0^x I_1(\alpha)\sin(x-\alpha)\frac{d\alpha}{\alpha}=I_0(x)-\cos x\,, \quad \ldots\ldots \quad (22)$$

$$\int_0^x I_1(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha}=\sin x-I_1(x)\,, \quad \ldots\ldots \quad (23)$$

$$\int_0^x I_2(\alpha)\cos(x-\alpha)\frac{d\alpha}{\alpha}=\tfrac{1}{2}\{I_0(x)-I_2(x)-\cos x\}\,, \quad \ldots \quad (24)$$

$$\int_0^x I_2(\alpha)\frac{d\alpha}{\alpha}=\tfrac{1}{2}\{1-I_0(x)-I_2(x)\}\,, \quad \ldots\ldots \quad (25)$$

$$\int_0^x I_4(a)\cos^2\tfrac{1}{2}(x-a)\frac{da}{a} = \tfrac{1}{4}\{\cos^2\tfrac{1}{2}x - I_0(x) - I_4(x)\}, \quad . \quad . \quad . \quad (26)$$

$$\int_0^x I_2(a)\cos^2\tfrac{1}{2}(x-a)\frac{da}{a} = \tfrac{1}{2}\{\sin^2\tfrac{1}{2}x - I_2(x)\}, \quad . \quad . \quad . \quad (27)$$

$$\int_0^x I_2(a)\sin^2\tfrac{1}{2}(x-a)\frac{da}{a} = \tfrac{1}{2}\{\cos^2\tfrac{1}{2}x - I_0(x)\}, \quad . \quad . \quad . \quad (28)$$

$$\int_0^x I_4(a)\sin^2\tfrac{1}{2}(x-a)\frac{da}{a} = \tfrac{1}{4}\{\sin^2\tfrac{1}{2}x - 2\,I_2(x)\}, \quad . \quad . \quad . \quad (29)$$

$$\int_0^x I_1(a)\sin a\frac{da}{a} = 1 - \cos x\, I_0(x) - \sin x\, I_1(x), \quad . \quad . \quad . \quad (30)$$

$$\int_0^x I_1(a)\cos a\frac{da}{a} = \sin x\, I_0(x) - \cos x\, I_1(x), \quad . \quad . \quad . \quad . \quad (31)$$

$$\int_0^x I_2(a)\sin a\frac{da}{a} = \tfrac{1}{2}\{\sin x\, I_0(x) - 2\cos x\, I_1(x) - \sin x\, I_2(x)\}, \quad . \quad (32)$$

$$\int_0^x I_2(a)\cos a\frac{da}{a} = \tfrac{1}{2}\{1 + \cos x\, I_0(x) + 2\sin x\, I_1(x) - \cos x\, I_2(x)\}. \quad (33)$$

Enfin la substitution $\varrho = -\frac{1}{2}$ dans (1) jusqu'à (7) nous donne, d'après $I_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$, les formules particulières:

$$2\sum_{n=0}^{\infty}(-1)^n I_{2n+\frac{3}{2}}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \sin(x-a)\sin a\frac{da}{a\sqrt{a}}, \quad . \quad . \quad (34)$$

$$\sum_{n=0}^{\infty}\varepsilon_{2n}(-1)^n I_{2n+\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \cos(x-a)\sin a\frac{da}{a\sqrt{a}}, \quad . \quad (35)$$

$$\sum_{n=0}^{\infty}\varepsilon_{2n} I_{2n+\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\sin a\,da}{a\sqrt{a}}, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\sum_{n=0}^{\infty} \varepsilon_{2n} I_{4n+\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \cos^2 \tfrac{1}{2}(x-a) \sin a \frac{da}{a\sqrt{a}}, \quad . \quad . \quad . \quad (37)$$

$$\sum_{n=0}^{\infty} I_{4n+\frac{5}{2}}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \sin^2 \tfrac{1}{2}(x-a) \sin a \frac{da}{a\sqrt{a}}, \quad . \quad . \quad . \quad (38)$$

$$2\sum_{n=0}^{\infty}(-1)^n \{\sin x\, I_{2n+\frac{1}{2}}(x) - \cos x\, I_{2n+\frac{3}{2}}(x)\} - \sqrt{\frac{2}{\pi x}} \sin^2 x = \frac{1}{\sqrt{2\pi}} \int_0^x \sin^2 a \frac{da}{a\sqrt{a}}, \quad . \quad . \quad (39)$$

$$2\sum_{n=0}^{\infty}(-1)^n \{\cos x\, I_{2n+\frac{1}{2}}(x) + \sin x\, I_{2n+\frac{3}{2}}(x)\} - \frac{1}{\sqrt{2\pi x}} \sin 2x = \frac{1}{2\sqrt{2\pi}} \int_0^x \sin 2a \frac{da}{a\sqrt{a}}. \quad (40)$$

§ 2. Eu égard aux séries

$$2\sum_{n=0}^{\infty}(-1)^n I_{2n+1}(x) = \sin x \quad \text{et} \quad 2\sum_{n=0}^{\infty} I_{4n+2}(x) = \sin^2 \tfrac{1}{2} x$$

nous substituons dans (a) $\varrho = 2n$ resp. $\varrho = 4n+1$, et ensuite nous multiplions les deux membres par $2(-1)^n$ resp. 2; après la sommation sur n de 0 à ∞ nous trouvons:

$$2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_{\nu+2n+1}(x) = \int_0^x I_\nu(x-a) \sin a \frac{da}{a},\ R(\nu) > -1 \quad . \quad (41)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} I_{\nu+4n+2}(x) = \int_0^x I_\nu(x-a) \sin^2 \tfrac{1}{2} a \frac{da}{a},\ R(\nu) > -1. \quad . \quad (42)$$

De celles-ci suivent pour $\nu = -\frac{1}{2}$ et $\nu = \frac{1}{2}$, d'après $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ et $I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, les formules particulières:

$$2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_{2n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos(x-a) \sin a \frac{da}{a\sqrt{x-a}}, \quad . \quad (43)$$

$$2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_{2n+\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin(x-a) \sin a \frac{da}{a\sqrt{x-a}}, \quad . \quad (44)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} I_{4n+\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos(x-a) \sin^2 \tfrac{1}{2} a \frac{da}{a\sqrt{x-a}}, \quad . \quad (45)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} I_{4n+\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin(x-\alpha) \sin^2 \tfrac{1}{2}\alpha \frac{d\alpha}{\alpha\sqrt{x-\alpha}} \quad . \quad . \quad (46)$$

Multiplions les deux membres de (43) et (44) par $\sin x$ resp. $\cos x$ et ceux de (45) et (46) par $\cos x$ resp. $\sin x$, alors nous trouvons par soustraction resp. addition des membres correspondants de chaque pair:

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \{ \sin x \, I_{2n+\frac{1}{2}}(x) - \cos x \, I_{2n+\frac{3}{2}}(x) \} = \sqrt{\frac{2}{\pi}} \int_0^x \sin^2 \alpha \frac{d\alpha}{\alpha\sqrt{x-\alpha}}, \quad (47)$$

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \{ \cos x \, I_{2n+\frac{1}{2}}(x) + \sin x \, I_{2n+\frac{3}{2}}(x) \} = \frac{1}{\sqrt{2\pi}} \int_0^x \sin 2\alpha \frac{d\alpha}{\alpha\sqrt{x-\alpha}}, \quad (48)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \{ \sin x \, I_{4n+\frac{3}{2}}(x) - \cos x \, I_{4n+\frac{3}{2}}(x) \} = \sqrt{\frac{2}{\pi}} \int_0^x \sin \alpha \sin^2 \tfrac{1}{2}\alpha \frac{d\alpha}{\alpha\sqrt{x-\alpha}}, \quad (49)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \{ \cos x \, I_{4n+\frac{3}{2}}(x) + \sin x \, I_{4n+\frac{5}{2}}(x) \} = \sqrt{\frac{2}{\pi}} \int_0^x \cos \alpha \sin^2 \tfrac{1}{2}\alpha \frac{d\alpha}{\alpha\sqrt{x-\alpha}}. \quad (50)$$

Enfin la substitution $\nu = 0$ dans (41) et (42) donne les formules particulières:

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_{2n+1}(x) = \int_0^x I_0(x-\alpha) \sin \alpha \frac{d\alpha}{\alpha}, \quad . \quad . \quad . \quad (51)$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} I_{4n+2}(x) = \int_0^x I_0(x-\alpha) \sin^2 \tfrac{1}{2}\alpha \frac{d\alpha}{\alpha}. \quad . \quad . \quad . \quad (52)$$

§ 3. Eu égard à la série connue [1]):

$$\sum_{n=0}^{\infty} (-1)^n (\varrho + 2n + 1) I_{\varrho+2n+1}(x) = \frac{x}{2} I_\varrho(x) \quad . \quad . \quad . \quad . \quad (I)$$

nous remplaçons dans (a) ϱ par $\varrho + 2n$ et ensuite nous multiplions les deux membres par $2(-1)^n(\varrho + 2n + 1)$; après la sommation sur n de 0 à ∞ nous trouvons:

$$2 \sum_{n=0}^{\infty} (-1)^n I_{\nu+\varrho+2n+1}(x) = \int_0^x I_\nu(x-\alpha) I_\varrho(\alpha) \, d\alpha, \quad \begin{cases} R(\nu) > -1 \\ R(\varrho) > -1 \end{cases} . \quad (53)$$

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[1]) NIELSEN, Handbuch der Theorie der Cylinderfunktionen 1904, p. 270, (3).

ou en substituant $\nu = \mu - \varrho - 1$:

$$2 \sum_{n=0}^{\infty} (-1)^n I_{\mu+2n}(x) = \int_0^x I_{\mu-\varrho-1}(x-a)\, I_\varrho(a)\, da, \quad R(\mu) > R(\varrho) > -1. \tag{54}$$

Comme des formules particulières suivent de (54) pour $\varrho = -\frac{1}{2}$ et $\varrho = \frac{1}{2}$:

$$\sum_{n=0}^{\infty} (-1)^n I_{\mu+2n}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x I_{\mu-\frac{1}{2}}(x-a) \cos a \frac{da}{\sqrt{a}}, \quad R(\mu) > -\tfrac{1}{2} \tag{55}$$

$$\sum_{n=0}^{\infty} (-1)^n I_{\mu+2n}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x I_{\mu-\frac{3}{2}}(x-a) \sin a \frac{da}{\sqrt{a}}, \quad R(\mu) > \tfrac{1}{2} \tag{56}$$

ou en remplaçant dans (56) μ par $\mu + 1$:

$$\sum_{n=0}^{\infty} (-1)^n I_{\mu+2n+1}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x I_{\mu-\frac{1}{2}}(x-a) \sin a \frac{da}{\sqrt{a}}, \quad R(\mu) > -\tfrac{1}{2}, \tag{56'}$$

Particulièrement la substitution de $\mu = \frac{1}{2}$ dans (54) et (55), et de $\mu = \frac{3}{2}$ dans (54) et (56) donne:

$$\left.\begin{aligned} \sum_{n=0}^{\infty} (-1)^n I_{2n+\frac{1}{2}}(x) &= \tfrac{1}{2} \int_0^x I_{-\varrho-\frac{1}{2}}(x-a)\, I_\varrho(a)\, da = \\ &= \frac{1}{\sqrt{2\pi}} \int_0^x I_0(x-a) \cos a \frac{da}{\sqrt{a}}, \quad \tfrac{1}{2} > R(\varrho) > -\tfrac{1}{2} \end{aligned}\right\} \tag{57}$$

$$\left.\begin{aligned} \sum_{n=0}^{\infty} (-1)^n I_{2n+\frac{3}{2}}(x) &= \tfrac{1}{2} \int_0^x I_{-\varrho+\frac{1}{2}}(x-a)\, I_\varrho(a)\, da = \\ &= \frac{1}{\sqrt{2\pi}} \int_0^x I_0(x-a) \sin a \frac{da}{\sqrt{a}}, \quad \tfrac{3}{2} > R(\varrho) > -\tfrac{1}{2} \end{aligned}\right\} \tag{58}$$

En multipliant les deux membres de (55) par $\sin x$ resp. $\cos x$ et ceux de (56′) par $\cos x$ resp. $\sin x$ on trouve par soustraction resp. addition des membres correspondants:

$$\sum_{n=0}^{\infty} (-1)^n \{\sin x\, I_{\mu+2n}(x) - \cos x\, I_{\mu+2n+1}(x)\} = \frac{1}{\sqrt{2\pi}} \int_0^x I_{\mu-\frac{1}{2}}(a) \sin a \frac{da}{\sqrt{x-a}}, \quad R(\mu) > -\tfrac{1}{2} \tag{59}$$

$$\sum_{n=0}^{\infty} (-1)^n \{\cos x\, I_{\mu+2n}(x) + \sin x\, I_{\mu+2n+1}(x)\} = \frac{1}{\sqrt{2\pi}} \int_0^x I_{\mu-\frac{1}{2}}(a) \cos a \frac{da}{\sqrt{x-a}}, \quad R(\mu) > -\tfrac{1}{2} \tag{60}$$

desquelles suivent pour $\mu=0$ et $\mu=1$ les formules particulières:

$$\sum_{n=0}^{\infty}(-1)^n \{\sin x\, I_{2n}(x) - \cos x\, I_{2n+1}(x)\} = \frac{1}{2\pi}\int_0^x \frac{\sin 2\alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}}, \qquad (61)$$

$$\sum_{n=0}^{\infty}(-1)^n \{\cos x\, I_{2n}(x) + \sin x\, I_{2n+1}(x)\} = \frac{1}{\pi}\int_0^x \frac{\cos^2 \alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}}, \qquad (62)$$

$$\sum_{n=0}^{\infty}(-1)^n \{\sin x\, I_{2n+1}(x) - \cos x\, I_{2n+2}(x)\} = \frac{1}{\pi}\int_0^x \frac{\sin^2 \alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}}, \qquad (63)$$

$$\sum_{n=0}^{\infty}(-1)^n \{\cos x\, I_{2n+1}(x) + \sin x\, I_{2n+2}(x)\} = \frac{1}{2\pi}\int_0^x \frac{\sin 2\alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}}. \qquad (64)$$

Après addition resp. soustraction des membres correspondants de (61) et (64) resp. (62) et (63) on trouve:

$$\int_0^x \frac{\sin 2\alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}} = \pi \sin x\, I_0(x), \quad . \ . \ . \ . \ . \ . \quad (65)$$

$$\int_0^x \frac{\cos 2\alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}} = \pi \cos x\, I_0(x), \quad . \ . \ . \ . \ . \ . \quad (66)$$

par conséquent:

$$\int_0^x \frac{\cos^2 \alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}} = \frac{\pi}{2}\{1+\cos x\, I_0(x)\}, \quad . \ . \ . \ . \quad (67)$$

$$\int_0^x \frac{\sin^2 \alpha\, d\alpha}{\sqrt{\alpha(x-\alpha)}} = \frac{\pi}{2}\{1-\cos x\, I_0(x)\}. \quad . \ . \ . \ . \quad (68)$$

Alors (61) jusqu'à (64) se réduisent à:

$$\left.\begin{aligned} &\sum_{n=0}^{\infty}(-1)^n\{\sin x\, I_{2n}(x) - \cos x\, I_{2n+1}(x)\} = \\ &\quad = \sum_{n=0}^{\infty}(-1)^n\{\cos x\, I_{2n+1}(x) + \sin x\, I_{2n+2}(x)\} = \tfrac{1}{2}\sin x\, I_0(x), \end{aligned}\right\} . \qquad (69)$$

$$\sum_{n=0}^{\infty}(-1)^n \{\cos x\, I_{2n}(x) + \sin x\, I_{2n+1}(x)\} = \tfrac{1}{2}\{1+\cos x\, I_0(x)\}, \qquad (70)$$

$$\sum_{n=0}^{\infty}(-1)^n \{\sin x\, I_{2n+1}(x) - \cos x\, I_{2n+2}(x)\} = \tfrac{1}{2}\{1-\cos x\, I_0(x)\}. \qquad (71)$$

En posant dans (54), (55) et (56′) successivement $\mu = 2m+1$ et $\mu = 2m$ (m entier positif ou nulle) on trouve, d'après les séries (b) et (c) et après changement des deux membres:

$$\int_0^x I_{2m-\varrho}(x-\alpha)\, I_\varrho(\alpha)\, d\alpha = (-1)^m \Big\{ \sin x - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n+1}(x) \Big\}, \; 2m+1 > R(\varrho) > -1 \quad (72)$$

$$\int_0^x I_{2m-\varrho-1}(x-\alpha)\, I_\varrho(\alpha)\, d\alpha = (-1)^m \Big\{ \cos x + I_0(x) - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n}(x) \Big\}, \; 2m > R(\varrho) > -1 \quad (73)$$

$$\int_0^x I_{2m+\frac{1}{2}}(x-\alpha) \cos\alpha \frac{d\alpha}{\sqrt{\alpha}} = (-1)^m \sqrt{\frac{\pi}{2}} \Big\{ \sin x - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n+1}(x) \Big\}, \quad m \geqq 0 \quad (74)$$

$$\int_0^x I_{2m-\frac{1}{2}}(x-\alpha) \cos\alpha \frac{d\alpha}{\sqrt{\alpha}} = (-1)^m \sqrt{\frac{\pi}{2}} \Big\{ \cos x + I_0(x) - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n}(x) \Big\}, \quad m \geqq 0 \quad (75)$$

$$\int_0^x I_{2m+\frac{1}{2}}(x-\alpha) \sin\alpha \frac{d\alpha}{\sqrt{\alpha}} = (-1)^m \sqrt{\frac{\pi}{2}} \Big\{ I_0(x) - \cos x - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n+2}(x) \Big\}, \quad m \geqq 0 \quad (76)$$

$$\int_0^x I_{2m-\frac{1}{2}}(x-\alpha) \sin\alpha \frac{d\alpha}{\sqrt{\alpha}} = (-1)^m \sqrt{\frac{\pi}{2}} \Big\{ \sin x - 2 \sum_{n=0}^{m-1} (-1)^n I_{2n+1}(x) \Big\}, \quad m \geqq 0. \quad (77)$$

Multiplions les deux membres de (74) par $\sin x$ resp. $\cos x$ et ceux de (76) par $\cos x$ resp. $\sin x$, nous trouvons par soustraction resp. addition des membres correspondants:

$$\left.\begin{aligned} &\int_0^x I_{2m+\frac{1}{2}}(\alpha) \sin\alpha \frac{d\alpha}{\sqrt{x-\alpha}} = \\ &= (-1)^m \sqrt{\frac{\pi}{2}} \Big[1 - \cos x\, I_0(x) + 2 \sum_{n=0}^{m-1} (-1)^n \{ \cos x\, I_{2n+2}(x) - \sin x\, I_{2n+1}(x) \}, \quad m \geqq 0 \end{aligned}\right\} (78)$$

$$\left.\begin{aligned} &\int_0^x I_{2m+\frac{1}{2}}(\alpha) \cos\alpha \frac{d\alpha}{\sqrt{x-\alpha}} = \\ &= (-1)^m \sqrt{\frac{\pi}{2}} \Big[\sin x\, I_0(x) - 2 \sum_{n=0}^{m-1} (-1)^n \{ \sin x\, I_{2n+2}(x) + \cos x\, I_{2n+1}(x) \}, \quad m \geqq 0. \end{aligned}\right\} (79)$$

En faisant de même à l'égard de (75) et (77) on obtient:

$$\left.\begin{aligned}&\int_0^x I_{2m-\frac{1}{2}}(\alpha)\sin\alpha\,\frac{d\alpha}{\sqrt{x-\alpha}}=\\&=(-1)^m\sqrt{\frac{\pi}{2}}\Big[\sin x\,I_0(x)+2\sum_{n=0}^{m-1}(-1)^n\{\cos x\,I_{2n+1}(x)-\sin x\,I_{2n}(x)\},\ m\geqq 0\end{aligned}\right\}\quad .\quad (80)$$

$$\left.\begin{aligned}&\int_0^x I_{2m-\frac{1}{2}}(\alpha)\cos\alpha\,\frac{d\alpha}{\sqrt{x-\alpha}}=\\&=(-1)^m\sqrt{\frac{\pi}{2}}\Big[1+\cos x\,I_0(x)-2\sum_{n=0}^{m-1}(-1)^n\{\sin x\,I_{2n+1}(x)+\cos x\,I_{2n}(x)\},\ m\geqq 0.\end{aligned}\right\}\quad .\quad (81)$$

La substitution $m=0$ dans (72) jusqu'à (77) donne les formules particulières:

$$\int_0^x I_\varrho(x-\alpha)\,I_{-\varrho}(\alpha)\,d\alpha=\sin x,\quad 1>R(\varrho)>-1\quad .\ .\ .\ .\quad (82)$$

$$\int_0^x I_\varrho(x-\alpha)\,I_{-\varrho-1}(\alpha)\,d\alpha=\cos x+I_0(x),\quad 0>R(\varrho)>-1\ .\quad (83)$$

$$\int_0^x \sin(x-\alpha)\cos\alpha\,\frac{d\alpha}{\sqrt{\alpha(x-\alpha)}}=\int_0^x\cos(x-\alpha)\sin\alpha\,\frac{d\alpha}{\sqrt{\alpha(x-\alpha)}}=\frac{\pi}{2}\sin x,\quad (84)$$

$$\int_0^x \cos(x-\alpha)\cos\alpha\,\frac{d\alpha}{\sqrt{\alpha(x-\alpha)}}=\frac{\pi}{2}\{I_0(x)+\cos x\},\quad .\ .\quad (85)$$

$$\int_0^x \sin(x-\alpha)\sin\alpha\,\frac{d\alpha}{\sqrt{\alpha(x-\alpha)}}=\frac{\pi}{2}\{I_0(x)-\cos x\}.\ .\ .\ .\quad (86)$$

La soustraction des deux premiers membres de (84) et la sommation des membres correspondants de (85) et (86) donne:

$$\int_0^x \frac{\sin(x-2\alpha)\,d\alpha}{\sqrt{\alpha(x-\alpha)}}=0\quad .\ .\ .\ .\ .\ .\ .\ .\quad (87)$$

$$\int_0^x \frac{\cos(x-2\alpha)\,d\alpha}{\sqrt{\alpha(x-\alpha)}}=\pi\,I_0(x)\quad .\ .\ .\ .\ .\ .\ .\quad (88)$$

La substitution $m=0$ dans (78) jusqu'à (81) donne des résultats déjà connus, nommément (68), (65) et (67).

Chemistry. — *Ion Activities in Suspensions (Preliminary note)*. By R. LOOSJES and A. C. SCHUFFELEN. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of March 29, 1941.)

It is of great interest to theoretical as well as practical chemistry that activities of ions bound to substances in colloidal dispersion be known: to theoretical chemistry, because the attracting forces of the colloidal surface can be calculated from the activities; to practical chemistry, because the influence of ions on numerous processes is a function of their activity. First of all we think of the hydrogen ion activity (pH) in which many a branch of science and technics is deeply interested. For example, one of the first determinations carried out during an investigation in fertility of a soil (containing colloidal clay and humus) being that of pH. However, it is known that other ions are also influencing plant growth and there is a possibility of the direct influence of these ions on plant growth being determined by their activities. But first of all, before we can test this supposition the activities of $Ca^{\cdot\cdot}$, $Mg^{\cdot\cdot}$, $K^{\cdot}$, $Na^{\cdot}$, $NH_4^{\cdot}$, NO_3', PO_4''' etc. are to be known. This is one example out of many others showing that it is desirable to know the activities of all ions in a suspension. In our case it was the very stimulation to make a research after a practical method of determining the activities of all these ions.

One could ask if the simplest way would not be to make a centrifugate or an ultrafiltrate and to determine concentrations of the requested ions in the clear solution correcting with the activity coefficient. WIEGNER and PALLMANN (1) however extensively proved the hydrogen ion activity of a suspension to be different from the appertaining ultrafiltrate. The adsorbed ions too are active. For some colloids this fact was already known to workers in the field of the DONNAN membrane equilibrium (2). We are thus compelled to determine the activity of other ion species also in suspension.

The electrometric method is certainly the best one. A specific electrode for every ion species is needed. Though for hydrogen and a few other ions there being a useful electrode at hand, yet their is none for the alkaline and earthalkaline metals. An exception may be made for the calcium electrodes of TENDELOO (3).

Beside the electrometric method there is the colorimetric one. This is only useful in clear solutions.

There are some semiquantitative methods for determining ion activity (or, as it is called in soil science: mobility or availability): the fractionation of ions by electrodialysis or exchange. Perhaps the determination by the

reactions of an organism (plant or fungus) belongs to this class, but this is still to be proved. By these methods the activity is determined only approximately, for the measured quantities are correlating with the activity in a way not yet known.

Now there is an indirect method of determining ion activities based on the second law of thermodynamics. From the investigations of WIEGNER and PALLMANN and many others we know there is a difference in pH between suspension and equilibrium solution (clear solution in equilibrium with the suspension). For a system in thermodynamical equilibrium we know that if the activity of an ion species is different at two different points there must be an electrical potential difference between these points (other variables such as temperature and pressure being the same). According to a well-known relation we may write:

$$E = \pm \frac{RT}{nF} \ln \frac{a_{i\ \mathrm{solution}}}{a_{i\ \mathrm{suspension}}} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (I)$$

(+ and — for positive and negative ions respectively).

E means the electrical potential difference, R the gas constant, T the absolute temperature, a_i the activity of the ion species *i*, n the valence of ions *i* and F is 96500 Coulombs. This equation holds for every ion species present. $a_{i\ \mathrm{solution}}$ can be determined by a quantitative analysis of the equilibrium solution correcting if necessary with the activity coefficient. E can be determined by introducing saturated-KCl calomel electrodes in solution and suspension. Now, $a_{i\ \mathrm{suspension}}$ is calculated by means of equation I. This method was tested by some workers (4) on the DONNAN membrane equilibrium, which is a special case of unequal ionic distribution (see BOLAM (5)).

The direct determination of E which would require a special apparatus is not necessary. For, if $a_{i\ \mathrm{solution}}/a_{i\ \mathrm{suspension}}$ is determined electrometrically for one ion species this quantity is known for every ion species present, E being of course the same in all cases. As the electrometrically determinable ion the hydrogen ion is very useful since it is present in all aqueous solutions. In this case we can use an equation derived from I:

$$pH_{\mathrm{susp.}} - pH_{\mathrm{sol.}} = \frac{pI^{\cdot}_{\mathrm{susp.}} - pI^{\cdot}_{\mathrm{sol.}}}{n_{i^{\cdot}}} = \frac{pI'_{\mathrm{sol.}} - pI'_{\mathrm{susp.}}}{n_{i'}} \quad . \ . \quad (II)$$

where $I^{\cdot}$ is a symbol for some cation and I' for some anion. This relation has been tested for anions only by some investigators on the DONNAN membrane equilibrium (6). They found it to be true.

We have used this formula II in determining activities of cations in a suspension. The procedure to be followed is very simple: a colloid is shaken with a quantity of water until equilibrium is attained. Next the equilibrium solution is separated in some way from the colloidal material, the pH's in both the colloidal suspension and the equilibrium solution are determined

in the usual way, and the equilibrium solution is analysed. In most cases the equilibrium solution being dilute the concentration found may be put equal to the activity. For the use we wanted to make (see later) of the activities an accuracy of ± 0.1 "pH unity" is sufficient. Thus, activity coefficients down to 0.9 may be safely neglected. Otherwise these may be calculated with the aid of formulae available (7). With the aid of formula II $pI_{suspension}$ is then to be calculated from the determined quantities.

Our experiments were carried out with the colloidal material called "*Dusarit*" (8). This is a sulfuric acid treated charcoal with a very high cation exchange capacity (± 2 mg aeq/g). Nevertheless its suspensions are so coarse that they are sedimenting within a short interval. We have chosen this material, being familiar with it because it is used in our laboratory as an artificial soil colloid for plant growth experiments (9). In connexion with these experiments we wanted to know the activity of the different ions.

First of all we had to test the usefulness of the method. Now silver is an ion that is readily determined electrometrically. By means of this ion a check of the activities found by our method is possible. We wanted to know the usefulness under different circumstances and so we carried out the following experiments.

1. By percolation with salt or acid solutions we prepared Ag and H-dusarits (that is: dusarit with Ag or H adsorbed). Attention is called to the fact, that the Ag-dusarits also contain H-ions. The dusarits were thoroughly washed with water to free them from electrolytes and dried. Different mixtures of dry Ag and H-dusarits were weighed out in portions of 5 g in total. These were mixed with 200 cm^3 0.001 n $NaNO_3$ solution, well shaken and put away. From time to time the suspensions were shaken again. $NaNO_3$ solution was used instead of pure water as we wanted to have a well defined quantity of anions in the equilibrium solution. After at least one night the supernatant liquid is decanted and filtered by suction through dry Jena IG4 glass filters to remove small floating particles. 100 cm^3 of this solution was analysed quantitatively. As the quantities to be analysed were small, we had to make use of a micromethod. Now, we had a quantitative flame spectrograph at hand and, as one of us recently proved (10), this apparatus yields sufficiently accurate results. So we spectrographed all our solutions. In a sample of the solution pH was determined by the glass electrode (with a Coleman 3D electrometer). Moreover pAg was determined electrometrically in this sample as a control on the spectrographical analysis. In the appertaining suspension, or better, in the sedimented mass pH and pAg were determined again. For measurements of pAg an Ag-AgI-electrode (electrolytically prepared) was used. This electrode is giving accurate results down to concentrations of 10^{-4} n without previously saturating with AgI, as we tested with $AgNO_3$ solutions of known activity (activity coefficients of MAC INNES and BROWN (11)). However, the Ag-AgI-electrode produced deviations unexplicated until

now after use in suspensions, when tested afterwards. Hereby deviations of 0.2 "pAg-unity" are possible. Reference electrode for Ag-AgI-electrode as well as glass electrode was a saturated-KCl calomel electrode, connexion between the two half elements being made by a 1.7 n KNO_3 and 0.3 n $NaNO_3$ agar bridge.

In table 1 results of these measurements and calculations are presented. A sufficient agreement between measured and calculated activities of silver is to be seen. It is remarkable that the activities of silver are of the same or smaller order than those of hydrogen. When calculating activity coefficients from these informations, the concentration of the Ag-ions in the suspension should be known. Now, concentration in a sedimented suspension is a rather difficult thing to determine. We have therefore adopted as a measure for the activity coefficient the quotient:

$$A = \frac{\text{activity}}{\dfrac{\text{mg aeq Ag}}{\text{g dusarit}}}$$

TABLE 1.

5 g Dusarit + 200 cm³ 0.001 n $NaNO_3$ solution.

Number	mg aeq Ag pro gram dusarit	Equilibrium solution				Sedimented suspension						
		Observed spectrographically		Observed electrometrically		Observed electrometrically		Computed from:				
								Spectrographical data		Electrometrical data	Spectrographical data	
		pNa	pAg	pAg	pH	pH	pAg	pAg	pNa	pAg	$A_{Ag} \times 10^3$	$A_{Na} \times 10^3$
1	1.11	4.2	3.1	3.1	3.1	1.6	1.7	1.6	2.8	1.7	22	47
7	0.95	3.9	3.3	3.3	3.1	1.6	1.8	1.8	2.4	1.8	17	110
2	0.89	4.0	3.2	3.3	2.8	1.4	1.9	1.8	2.6	1.9	17	69
12	0.84	3.9	3.3	3.5	3.1	1.8	2.0	2.1	2.6	2.2	10	66
8	0.76	3.9	3.4	3.5	2.9	1.4	2.0	1.9	2.5	2.0	15	87
3	0.67	4.1	3.4	3.6	2.6	1.2	2.2	2.1	2.7	2.2	14	62
9	0.57	4.0	3.6	3.8	2.7	1.3	2.2	2.1	2.6	2.4	13	74
13	0.50	—	3.6	3.6	2.7	1.6	2.3	2.5	—	2.5	6	—
4	0.44	4.1	3.6	3.7	2.4	1.2	2.5	2.4	2.9	2.5	9	31
10	0.38	4.0	3.8	3.9	2.6	1.2	2.6	2.4	2.6	2.5	10	63
5	0.22	4.2	4.1	4.2	2.4	1.2	2.9	2.8	3.0	3.0	7	28
11	0.19	4.1	4.4	4.2	2.5	1.3	3.1	3.2	2.9	3.0	3	34
14	0.17	4.5	4.2	4.0	2.5	1.5	3.0	3.2	3.5	3.0	4	8

The denominator is linearly proportional to the concentration provided the packing of the sedimented suspension is always equally tight. If this condition is fulfilled (it need not always be so), then the quotient will be linearly proportional to the activity coefficient. In table 1 this quotient A for Ag and Na is also presented. The error of A is $\pm$ 20 %. There is a distinct increase of A at increasing Ag concentrations. The values of A_{Ag} are much lower than those of A_{Na}, indicating a stronger binding force for Ag-ions than for Na-ions.

2. Determinations were also carried out under more complicated circumstances. Mixtures of Ag, Na and H-dusarits were made, the quantity of Ag being constant, the ratio of adsorbed Na and H changing, thus lowering the H-concentration at constant Ag-concentration. The Na-dusarit was prepared in the same way as the Ag-dusarit. In table 2 results are given. Of course, pNa cannot be checked. It is seen that changing the Na/H ratio has a great influence on the activity of the Ag-ions.

TABLE 2.

5 g Dusarit + 200 cm^3 0.001 n $NaNO_3$ solution.

Number	mg aeq Ag pro gram dusarit	mg aeq Na pro gram dusarit	Equilibrium solution				Sedimented suspension						
			Observed spectrographically		Observed electrometrically		Observed electrometrically		Computed from:				
									Spectrographical data		Electrometrical data	Spectrographical data	
			pNa	pAg	pAg	pH	pH	pAg	pAg	pNa	pAg	$A_{Ag} \times 10^3$	$A_{Na} \times 10^3$
20	0.19	1.66	2.8	4.8	5.4	6.5	5.2	3.9	3.5	1.5	4.1	2	21
21	0.19	1.25	2.8	4.5	4.8	4.8	3.7	3.4	3.3	1.6	3.7	3	19
22	0.19	0.83	2.8	4.1	4.7	3.3	2.1	3.1	2.9	1.6	3.5	6	32
23	0.19	0.42	3.2	4.0	4.2	2.8	1.5	2.9	2.7	1.7	2.9	12	42
24	0.19	—	3.8	4.0	4.2	2.5	1.1	2.9	2.5	2.4	2.8	15	118

3. In mixtures of Ag, Ca and H-dusarits activities were determined, the quantity of Ag-dusarit being constant, the ratio of adsorbed Ca and H changing. This experiment is of particular interest to soil science as in soils the ratio of adsorbed Ca and H is one of the mightiest factors governing plant growth. The Ca-dusarit was prepared in the same way as the Ag-dusarit. In table 3 results of these experiments are presented. In calculating pCa we must pay attention to the fact that $n = 2$. The maximum to be seen in the values of A_{Ag} (or the minimum in pAg) and in a slighter degree in A_{Na} and A_{Ca} may be very significant for the plant growth in soil.

TABLE 3.

5 g Dusarit + 200 cm^3 0.001 n $NaNO_3$ solution.

Number	mg aeq Ag pro gram dusarit	mg aeq Ca pro gram dusarit	Equilibrium solution: Observed spectrographically			Equilibrium solution: Observed electrometrically		Sedimented suspension: Observed electrometrically		Sedimented suspension, Computed from: Spectrographical data			Computed from: Electrometrical data	Computed from: Spectrographical data	
			pNa	pCa	pAg	pAg	pH	pH	pAg	pAg	pNa	pCa	pAg	$A_{Ag} \times 10^3$	$A_{Na} \times 10^3$
15	0.22	1.66	3.2	3.5	3.7	3.8	5.4	5.0	3.3	3.3	2.8	2.7	3.4	3	115
16	0.22	1.24	3.2	3.6	3.3	3.5	3.9	3.4	2.8	2.8	2.7	2.6	3.0	8	115
17	0.22	0.83	3.4	4.4	3.4	3.6	2.9	2.0	2.6	2.5	2.5	2.7	2.8	16	129
18	0.22	0.41	3.8	5.5	3.8	4.0	2.6	1.6	2.9	2.8	2.8	3.5	3.0	8	49
19	0.22	—	4.1	—	4.0	4.3	2.5	1.6	3.2	3.0	3.2	—	3.3	5	18

4. Though a perceptible influence of the small quantity of ions given off to the solutions on the p's found, was not probable, some experiments were carried out to confirm this supposition. Several portions of two different Ag and H-dusarit mixtures were mixed with a constant quantity of $NaNO_3$ solved in changing quantities of water. pAg and pNa were determined in the usual ways. Results are given in table 4. Dilution is not reflected in the values found.

TABLE 4.

5 g Dusarit + 200 cm^3 0.001 n $NaNO_3$ solution + a cm^3 water.

Number	mg aeq Ag pro gram dusarit	cm^3 added (a)	Equilibrium solution: Observed spectrographically		Equilibrium solution: Observed electrometrically		Sedimented suspension: Observed electrometrically		Sedimented suspension, Computed from: Spectrographical data		Computed from: Electrometrical data	Computed from: Spectrographical data	
			pNa	pAg	pAg	pH	pH	pAg	pAg	pNa	pAg	$A_{Ag} \times 10^3$	$A_{Na} \times 10^3$
25	0.17	0	3.9	4.2	4.2	2.6	1.5	2.9	3.1	2.8	3.1	5	44
26	0.17	100	4.3	4.3	4.4	2.7	1.5	3.4	3.1	3.1	3.2	5	21
27	0.17	200	4.1	4.4	4.3	2.9	1.6	3.1	3.1	2.9	3.1	5	36
28	0.34	0	3.9	3.7	4.0	2.6	1.7	2.8	2.8	3.0	3.1	5	32
29	0.34	200	4.4	4.1	4.1	2.9	1.6	2.8	2.8	3.1	3.0	5	20

5. For substances that are not separated from their equilibrium solution as easily as dusarit, an other method has been tried. In this case mixtures of dusarits were dialysed in little collodion bags closed with a paraffined

cork, against the 200 cm^3 0.001 n $NaNO_3$ solution, by rotating solutions and bags in a 300 cm^3 Jena wide-mouth flask during 32 hours. All treatments were performed in the dark or in red light in order to prevent reduction of silver in the presence of collodion. Results are presented in table 5 showing a satisfactory agreement between pAg measured directly and indirectly.

TABLE 5.

5 g Dusarit + 200 cm^3 0.001 n $NaNO_3$ solution.

Number	mg aeq Ag pro gram dusarit	Equilibrium solution				Sedimented suspension						
		Observed spectrographically		Observed electrometrically		Observed electrometrically		Computed from:				
								Spectrographical data		Electrometrical data	Spectrographical data	
		pNa	pAg	pAg	pH	pH	pAg	pAg	pNa	pAg	$A_{Ag} \times 10^3$	$A_{Na} \times 10^3$
30	0.95	3.8	3.2	3.2	3.0	1.8	2.1	2.0	2.6	2.0	10	95
31	0.67	3.9	3.4	3.4	2.8	1.7	2.3	2.3	2.8	2.3	8	45
32	0.50	4.0	3.5	3.6	2.7	1.5	2.4	2.4	2.8	2.5	9	41
33	0.34	4.1	3.8	3.9	2.6	1.5	2.7	2.7	3.0	2.8	7	31

At the end of this article it is tempting to draw further conclusions from the results presented. However, we are desisting from it in view of the still small number of experiments performed. We only wanted to point at this useful method for the determination of ion activity in suspensions and to show its rightness, just indicating the influence of the different cation species on each other's activity.

Summary.

A simple procedure is given according to which it is possible to determine activities in suspension of ions not measureable with a specific electrode. The method was tested under different circumstances by determining the silver ion activity also electrometrically. Results were satisfactory. The method was applied to the determination of calcium and sodium ion activity. Experiments are demonstrating the influence of the different ions on the activities and the "activity coefficients" of each other. "Dusarit" was used as colloidal material.

The authors wish to express their thanks to Prof. Ir J. HUDIG for giving them the opportunity to work out the above-mentioned method, to Prof. Dr H. R. KRUYT for his valuable criticism and Mr J. P. VISSER for his assistance with the analyses.

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Comparative Physiology. — *On the Life-History of Ammophila campestris Jur.* By G. P. BAERENDS. (Communicated by Prof. H. J. JORDAN.)

(Communicated at the meeting of March 29, 1941.)

The reproductive activities of the females of *Ammophila campestris* a digger wasp belonging to the *Sphecidae*, consists of coition, digging one cell nests, and capturing and paralysing caterpillars (and occasionally larva of Tenthredinidae), which they store in the nests as food for the larva. Observations made by ADLERZ (1903, 1909) in *Sweden* and by CRÈVECOEUR (1932) in *Belgium* have shown that *campestris* does not store all caterpillars before the hatching of the egg, as digger wasps usually do, but that she brings only one caterpillar before, and the rest after the larva has hatched. Moreover, some of ADLERZ' observations seem to suggest that *campestris* might, at least occasionally, be occupied with more than one nest at the same time.

These peculiarities had never been observed before in any digger wasp. If ADLERZ' suggestion were right, it would imply such a highly organised behaviour as could hardly be expected in any insect. The wasp would have to remember the localities of her different nests, not only for a single day, but sometimes for nearly a fortnight, for, during the nights and bad weather periods, the wasps stay out in the heather and do not visit the nests. In the Dutch climate bad weather may continue for some weeks.

In addition, there must be some regulatory system that guarantees an equal food distribution among the different nests; presumably the wasp can be stimulated somehow to start or to stop provisioning a certain nest.

These questions induced me to make a closer study of the life-history of this species.

To get an exact knowledge of the course of events of the wasp's activities, it was in the first place necessary to observe individual wasps during a series of consecutive days. Individual identification of each wasp was possible by a system of coloured markings.

Twenty wasps were continuously watched during about a fortnight each. My observations of these wasps cover the activities at the nesting places as completely as possible. I did not, however, follow them on their foraging and hunting excursions in the heather. As I could be sure, however, that none of the marked wasps had a nest outside the observed area, I can assume that none of their nesting activities escaped my attention.

Here follows a brief account of the most important results of these observations.

During the first bright days in June the males leave the cocoons. The

females appear some days later. Mating takes place within a few hours after the female has hatched. Shortly after coition the female begins to dig her first nest (nest A). As nesting sites flat, sandy areas with a compact soil are selected. The nest consists of a vertical shaft about 2½ cm long and one elliptical cell about 2 cm in length. Having finished the nest, the shaft is temporarily closed by loosely filling it up with some clods, and the wasp then disappears into the heather. She usually returns within a few hours, carrying a paralysed caterpillar. She reopens the nest, carries the caterpillar down and lays an egg. Then she closes the nest with much more care than before. After she has finished, it is impossible for the human eye to distinguish the entrance from the surroundings. She now leaves nest A and it may be some days, before she provisions it again.

Soon afterwards the wasp begins to dig a new nest (B).

It appeared to be a general rule that once a nest was begun, the work at this nest was continued without interruption, until an egg was laid.

The first series of activities, therefore, constituting the first stage of the care for every nest made, will be called the first phase.

After completing the first phase of nest B, which takes her one or more days, dependent on the amount of sunshine, the wasp returns to nest A and, before fetching a caterpillar, opens the nest. After a brief visit, she closes it again and flies off to the heather. Such a visit, in which no caterpillar is brought, will be called a "solitary visit", in contrast to "provisioning visit". As a rule the wasp, after this solitary visit brings one or more fresh caterpillars, before she leaves the nest alone for the second time. This phase I shall call, therefore, the second phase; it consists of a solitary visit followed by storing 1—3 caterpillars. Occasionally this phase consists of a solitary visit only, namely when the egg has not yet hatched at the time of this visit.

After having finished this second phase in nest A, the wasp carries through the same phase in nest B. Now she again pays a solitary visit to nest A and there enters into the third and last phase, consisting of one or more solitary visits and the storeage of another 3—7 caterpillars. This phase is concluded by closing the nest in an especially careful manner, the wasp pressing down the contents of the shaft with her head, during which a loud humming sound can be heard.

She now goes to nest B to accomplish the third phase here. After having finally closed this nest she begins to dig a new nest.

As we see from the above, in each nest provisioning occurs in three phases. During every phase the wasp is occupied with one particular nest exclusively, interrupting work at the nest only for foraging on her own behalf, for hunting caterpillars or for sleeping, but never for any work concerning another nest.

Having finished one phase she goes to another nest and works through an entire phase there. If there is no other nest to provide for, she makes a new nest. Occasionally, in very favourable weather, the wasp, after

having completed the second phase in nest A, digs a new nest C before beginning the second phase of nest B. In this way a wasp sometimes has three nests under her care.

The second and the third phase always begin with a solitary visit; sometimes a phase consists of even no more than that one solitary visit. This may occur when the nest has been disturbed shortly before this visit, or, in the case of the second phase, when the egg has not yet hatched. This suggests that the solitary visit serves as an inspection, that is to say that the wasp during this visit receives stimuli from the contents of the cell which determine whether she will leave the nest alone or go and fetch fresh caterpillars.

An experimental test of this hypothesis is possible on the following basis. If the solitary visit actually has a regulating function, it should be possible to influence the wasp's subsequent behaviour by changing the contents of the nest just before the solitary visit. This could not be done in the real nests but it appeared that the wasps did not interrupt their provisioning activities, when I replaced their nests by artificial nests, provided certain precautions were taken. These nests were made of gypsum and consisted of a lower part, containing the cell, and a lid that could easely be lifted so that I could reach the cell and change its contents at will.

I carried out the following experiments:

1. Nests, which according to preceding observations, should be provisioned immediately after the solitary visit, were disturbed by removing the larva just before that visit. The result was that the nest was abandoned after the first solitary visit.

2. In similar nests, I replaced the larva by a paralysed caterpillar with an *Ammophila's* egg, taken from another nest. Now the wasp did not start provisioning immediately after the solitary visit (as she should have done), but she waited until the egg was hatched.

3. Before the wasp paid her first solitary visit of the third phase to the nest I added some paralysed caterpillars to the contents of the nest. The result was that the wasp either stopped provisioning altogether, or at least brought less caterpillars than the smallest amount ever stored under normal conditions.

4. In nests containing one caterpillar with an *Ammophila's* egg, nests, therefore, that should not be provisioned immediately, I replaced the egg by a larva. In these cases the wasp brought fresh caterpillars soon after the solitary visit.

5. Occasionally a wasp pays a solitary visit when the third phase is halfway concluded. A few times I succeeded in taking all caterpillars away just before the visit. Normally the wasps should have brought only a few more caterpillars, but now they again stored a considerable number of caterpillars, making the total amount of stored food larger than ever observed under normal conditions.

The same experiments were carried out just before provisioning visits.

Under these circumstances the wasps did not react to any change in the contents of the cell. They even continued provisioning when the larva was removed together with the food.

These experiments conclusively show, therefore, that the solitary visit is a real inspection, during which the wasp learns how to act in the following hours or even days.

Apart from having a function as a regulatory principle, the solitary visit also demonstrates a most remarkable psychological fact: although the external stimulating situation is exactly the same at a solitary visit as at a provisioning visit, the wasp's behaviour is profoundly influenced by it at the first occasion, whereas at the second occasion not the slightest influence can be traced.

In one case, however, the contents of the nest does influence the wasp's behaviour during a provisioning visit. When I put certain objects into the cell just before the wasp would pay her very first visit, during which she had to lay her egg, the wasp did react. If the object was a caterpillar with an *Ammophila's* egg or a cocoon, the wasp pulled it out of the cell and threw it away. If it was a larva, she immediately brought in her caterpillar but failed to lay an egg. Often she even captured some more caterpillars and stored them, still postponing the laying of the egg. It appeared that the presence of a young larva stimulated the wasp to bring 1—3 caterpillars (corresponding with the second phase) and that an older larva stimulated the wasp to bring 3—7 caterpillars (corresponding with the third phase).

Whereas, as we have seen, it is, the amount of food present at the solitary inspection visit which determines the wasp's behaviour in the second and third phase, the wasp is stimulated at her first visit by the age of the larva.

Now the two series of experiments that served to investigate the part played by the solitary visit revealed the regulatory system, at work within the second or the third phase. They did not answer the questions as to the factors that bring the wasp from one phase into the next phase in the same nest.

Here the third series offers a suggestion. The age of the larva, which the wasp happens to find in her nest at her first visit, determines whether she will be brought into the second or into the third phase. This and other arguments, which cannot be treated in detail here, render it probable that the age of the larva has the same influence during the solitary visits.

The following may be illustrated by the narrative of the activities of wasp GO (fig. 1).

The wasp was marked shortly before she finished the third phase in nest A. After having closed this nest she paid a solitary visit to nest B. As the egg had not yet hatched, she closed this nest again without bringing a caterpillar and began to dig a new nest C. Next morning she brought in a caterpillar and laid an egg in this nest. Then she paid a solitary visit to B again where the larva had just hatched. Provisioning, therefore, is

started (third phase) and is continued for 2 days. After she has finally closed nest B, the wasp pays a solitary visit to C. The egg in C has not yet hatched whereupon the wasp leaves C alone and starts digging (D). As this was begun at a late hour, nest D was not completed before the end of the day. Next morning the wasp brought a solitary visit in C, nest D apparently being abandoned. In C, the larva has hatched and the

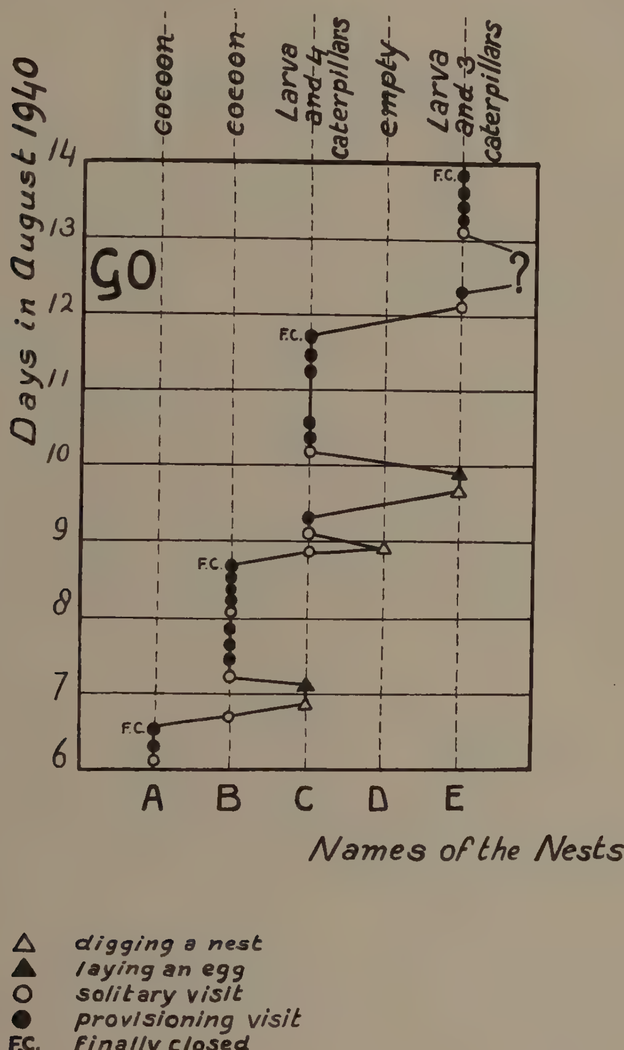

Fig. 1. Diagram of the activities of the wasp GO from August 6—14. All evidence about one nest is given in a vertical column at the top of which the condition of the brood on August 14 is given.

wasp brought one caterpillar. Then a new nest E was started, an egg was laid, whereupon, next day, the third phase in C is completed, which took 2 days. Next morning a solitary visit was paid to nest E where the larva had hatched. As a consequence one caterpillar was brought soon after-

wards. After that the wasp again began a new nest which is not taken into account here, because of incompleteness of my observations. Next day the third phase in nest E was completed.

Summary.

1. *Ammophila campestris Jur.* is able to look after 2 or even 3 nests at the same time.

2. The provisioning of each nest occurs in three phases, during each of which the wasp works at one nest exclusively.

The first phase consists of digging the nest, capturing and storing the first caterpillar and laying an egg. During the second phase 1—3 caterpillars are brought and during the third phase the wasp stores 3—7 caterpillars.

3. Both the second and the third phase begin with a solitary visit at which no caterpillar is brought.

4. Experiments show that these solitary visits have the function of an inspection, that is to say the quantity of food present at a solitary visit determines the wasp's subsequent provisioning behaviour, while absence of the larva causes the nest to be abandoned. During all provisioning visits except the first, the wasp is entirely insusceptible to the contents of the nest.

The behaviour of this species reveales a highly developed psychological organisation. The wasp must have an excellent memory for the different nest sites and, in addition, must be able to react to stimuli, received during an inspection-visit, which has taken place several hours or even more than a day before.

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Botany. — *On the development of the stellate form of the pith cells of Juncus species.* I. By R. A. MAAS GEESTERANUS. (Communicated by Prof. G. VAN ITERSON.)

(Communicated at the meeting of March 29, 1941.)

Introduction.

The remarkable stellate form of the pith cells in the so-called stem of *Juncus* species (the occurrence of such cells is by no means restricted to these plants) has attracted the attention of plant anatomists ever since TREVIRANUS (20). A survey of the principal literature of these remarkable star-shaped cells, which are different from most cells, are found in a very accurate study by F. T. LEWIS (12) 1925.

Theories have previously been advanced as to the causes leading to this shape. For them we refer to a survey in a publication by Miss L. M. SNOW (17) 1914. For our further observations the conception of S. SCHWENDENER (16) is of importance, who as far back as 1874 maintained that the surrounding tissue grows more rapidly than the pith, thus stretching its cells; under this influence the cells grow out, according to SCHWENDENER, to radiating cell arms to the side of the places where they touch each other. Miss SNOW adhered to this theory.

Detailed observations about the origin of star-shaped pith cells have also been published by A. ZIMMERMANN (22) in 1893 [1]). This investigator thought that the development of stellate cells from the meristematic tissue of cells hexagonal on transverse section (in which there are only small intercellular spaces near the vertices of the cells) could be explained by the supposition that parts of the cell walls to the side of the points of attachment became plastic and by the stretching of those parts under the influence of the turgor pressure. At the same time new cell wall material was supposed to be deposited in those parts through intussusception. With this theory ZIMMERMANN remained in agreement with the "turgor growth theory" formulated by him. Meanwhile ZIMMERMANN remarks that it is also conceivable that the surrounding growing tissue stretches the stellate pith cells mechanically and that this is the cause of the stretching of the cell wall near the points of attachment (according to him the wall near the intercellulars is thinner than the rest of the cell wall). The actual growing out, however, would still be brought about by intussusception in the parts of the wall stretched mechanically. This second possibility is nothing but the theory of SCHWENDENER, which is not mentioned by ZIMMERMANN.

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1) This publication has escaped the notice of Miss SNOW and of LEWIS.

In his work "On growth and form" (1917) D'ARCY WENTWORTH THOMPSON gave an explanation of the development of the star-shaped cells, which also agrees mainly with that of SCHWENDENER. Here D'ARCY THOMPSON starts from the supposition, afterwards opposed by LEWIS, that in a juvenile stage the pith tissue is comparable with an accumulation of globular cells according to "closest packing" of globes and he considers the expansion of the "boundary wall (that is the peripheral ring of woody and other tissues) which continues to expand after the pith cells which it encloses have ceased to grow or to multiply" as the causal force for the formation of the star-shaped cells: "The twelve points of attachment on the spherical surface of each little pith cell are uniformly drawn asunder".

LEWIS has rightly maintained, that the conception of D'ARCY W. THOMPSON concerning the construction of juvenile pith tissue is not correct and that such tissue should rather be compared with an accumulation of congruent cubo-octahedrons (tetrakai-decahedrals) with which the space can be completely filled. Intercellular spaces were supposed to form through cleavage of the intervening substance near the vertices and along the edges of the cell, followed by rounding off of these vertices and edges. LEWIS further assumes that the central parts of the lateral faces of adjacent cells remain attached. On subsequent growth of the pith the parts of the wall near the points of attachment are supposed to become the radii of the star-shaped cells. As in a pile of cubo-octahedrons any cubo-octahedron has walls in common with 14 other cubo-octahedrons, one might expect 14 radii. Actually 12 radii are usually found in the stellate pith cells which, as LEWIS showed, is a consequence of the fact that as a rule the horizontal upper and lower faces of the cubo-octahedral cell are detached from the corresponding faces of the cubo-octahedral cells situated above and below that cell, without forming cell arms.

Although in explanation of the formation of the typical star-shape of the pith cells LEWIS also ascribes some importance to the fact that the cells aim at occupying the minimal area, possible under the special circumstances, yet as causal force he assumes the same action as SCHWENDENER, ZIMMERMANN and D'ARCY WENTWORTH THOMPSON. This is very evident from a quotation by LEWIS from a letter of Prof. H. N. DAVIS, who made some mathematical calculations for LEWIS, namely: "It is necessary either that their surrounding should have grown too fast for them, thus pulling them away from each other, or that their internal volume should have shrunk very considerably after their distances apart had been determined by their early growth while still in tetrakaidecahedral form. In any case there must have been tension along the arms when the stellate form was attained."

It seemed possible that a study of the development of the stellate form of the pith cells of *Juncus*, in which, apart from the change in shape of the cells, the way of thickening and the optic properties of the cell walls were also taken into account, might throw fresh light on the development

of the remarkable shape of the pith cells. The results of such a study follow here.

1. *The material used for the investigation.*

The *Juncus* plants used belonged to the species of *effusus* L. They were collected in the autumn of 1939, with a clod of earth and placed in a cold frame till the moment when they were used. All the plants were sterile.

For the sake of orientation sections were made by hand. For the microtome sections the material was fixed in JUEL's fluid; it was embedded in paraffin with a melting point of 50° C. The sections were 5—10 μ thick and were stained according to the tannin-ferrichlorid method described in STRASSBURGER's "Botanisches Praktikum" p. 350.

It is desirable here to make some observations about the point at which we made the sections, because the construction of the so-called Juncus stem is often wrongly interpreted. A so-called sterile stem of the Juncus is namely not a stem, as is frequently mentioned in the literature, but should for the greater part be regarded as a leaf, round like a stem (in fertile stems

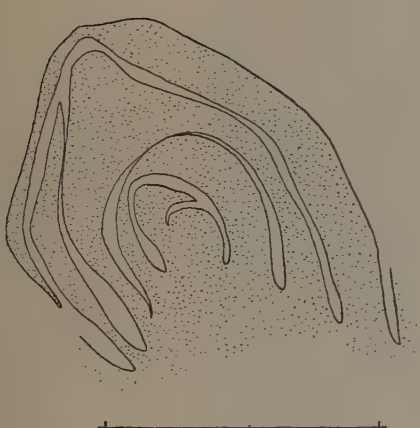

Fig. 1.

Growing point of *Juncus effusus* with young leaves in longitudinal section; the youngest leaf will develop into the circular leaf — the apparent stem.

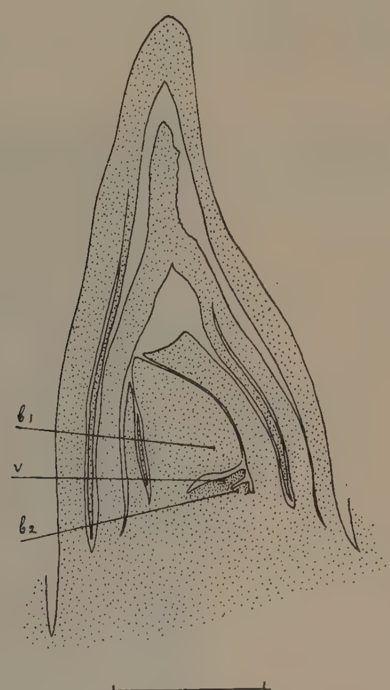

Fig. 2.

Growing point of *Juncus effusus* in longitudinal section at a slightly older stage than that of fig. 1; *b*1 and *b*2 are parts of what will later be the circular leaf, *v* is the flat vegetation point.

the part above the place of implantation of the inflorescence placed sideways should be taken as such a leaf, the part below it should there be considered the actual stem). For this matter we refer to TH. IRMISCH (9), M. LAURENT (11) and F. BUCHENAU (2).

This leaf, round like a stem, is implanted with its base on the very short, true stem and there the vegetation point of the stem is found. There the leaf mentioned overlaps the practically horizontal growing point and the very low space over this point is only connected with the outer air through a very small cleft on the base of the morphological anterior side of the leaf.

This remarkable situation becomes clear only through a study of the development of the cylindrical leaf. Here the reproduction of three stages in this development (figures 1, 2, and 3) may suffice. In the first, very juvenile stage (the shoot was only about 0.5 cm long) the leaf has grown like a cowl over the still curved point of vegetation. In the second stage the overgrowth (*b* 1) is even stronger, while the growing point, which for the rest remains undisturbed, is bounded by a practically horizontal plane. A little below the growing point we find another excretion (*b* 2), which also belongs to the cylindrical leaf, as the leaf entirely surrounds the growing point at its base; at first the leaf even

Fig. 3.

Photographical representation of the longitudinal section of the growing point of *Juncus effusus* at a slightly older stage than that of fig. 2, and in a plane perpendicular to that of fig. 2; the cylindrical body, which at the bottom has a fissure, is a juvenile stage of the circular leaf.

appears as a cylindrical covering at the periphery of the growing point. Between *b* 2 and the lower anterior side of *b* 1 there is the cleft mentioned, which connects the space above the growing point with the outer air.

In the third stage of development (shown in fig. 3) the cylindrical leaf has already attained some considerable size; it is seen as a parabolical body in the centre of the figure. The vertical section shown was made perpendicularly to the plane in which the section of fig. 2 was made. The hollow at the base of the cylindrical leaf is the space above the growing point, which is again bounded by a more or less horizontal surface.

Until this stage the growth of the cylindrical leaf was for the greater part effected through divisions of the meristem above the hollow mentioned; for the rest it will be caused exclusively by such divisions and by enlargement and transformation of the cells formed in this process. Hence the stellate pith cells mentioned originate from this meristem. The stellate form, however, arises only at a later stage than that of figure 3; stellate forms are found in shoots about 1 cm long.

The sections from which we studied the development of the pith were therefore made at various distances above the lens-shaped hollow mentioned before.

2. *The successive stages of development of the pith cells.*

Close above the lens-shaped space vaulting the growing point, the tissue is of an evidently meristematic nature. The cells of this tissue are practically entirely filled with protoplasm and possess a nucleus situated in the centre. As may be observed in fig. 4, showing a transverse section of this tissue,

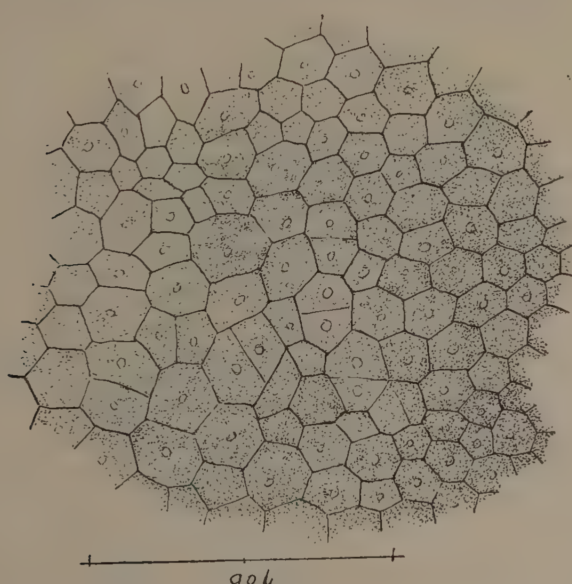

Fig. 4.
Transverse section through the meristem above the cleft-like hollow, bounding the flat growing point of *Juncus effusus*; here and there cell divisions.

several of these cells show divisions by division walls placed vertically, increasing the number of cells horizontally. Far more numerous, however, are divisions of these cells by horizontal walls, through which vertical rows of cells arise from initial cells (see fig. 12) causing the elongation of the cylindrical leaf. Hence at a juvenile stage the pith of this leaf consists of vertical rows of cells placed side by side.

On transverse as well as on longitudinal section the meristematic cells are polygonal, most of the cells being hexagonal on these sections. The diameter of the cells is small, averagely ca. 15 μ in a shoot 14 cm long. The size of the cells varies, however, according to the age of the plant.

We shall first consider the development of the pith cells as observed on the transverse sections. Here we note that it has previously been described by various botanists (SCHACH in 1856, DUCHARTRE, J. DUVAL JOUVE in 1869, A. ZIMMERMANN in 1893, D'ARCY W. THOMPSON 1917). Although, therefore, we can mention but little news in this respect we must yet touch on this development, in order to make intelligible what will follow subsequently.

A transverse section of a part of the meristem, which is at a somewhat later stage of development than the one we discussed above (this part is found only at about 0.1 mm above the one discussed) shows an early stage of the development of the intercellular spaces. This is depicted in fig. 5. We note that this figure is reproduced on a smaller scale than fig. 4; therefore in reality the cells of fig. 5 are considerably larger already than those of fig. 4. The intercellular spaces occur at the vertices of cells hexagonal on section. These spaces are triangular on these sections, the triangular sides curving in. At this stage the cells themselves are rounded off; it is very likely that — inversely — the occurrence of the intercellular spaces should be consider-

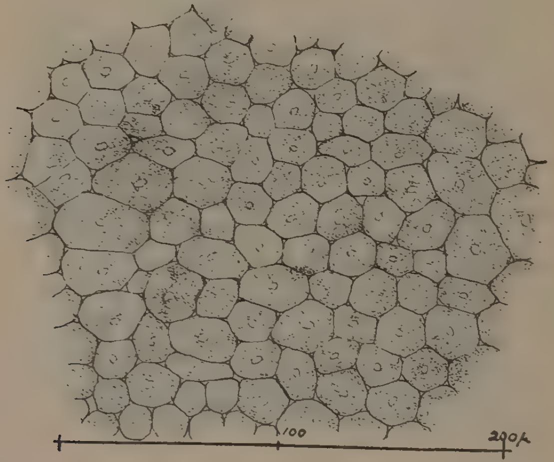

Fig. 5.

Transverse section through the meristem, about 0.1 mm higher than the section in fig. 4. It shows the occurrence of triangular intercellular spaces with the sides of the triangles curved inwards.

ed as a consequence of the tendency to rounding off on the part of the cells. This tendency is caused by the turgor pressure in the cells. This pressure will also cause the survival of points of attachment between the cells. Presumably the central lamellae near the vertices (and near the edges of the cells) are of a slightly different composition than those between the planes of attachment; possibly the long duration of the contact between the cells on those planes also plays a part.

A transverse section (see fig. 6) through a slightly older stage (ab.

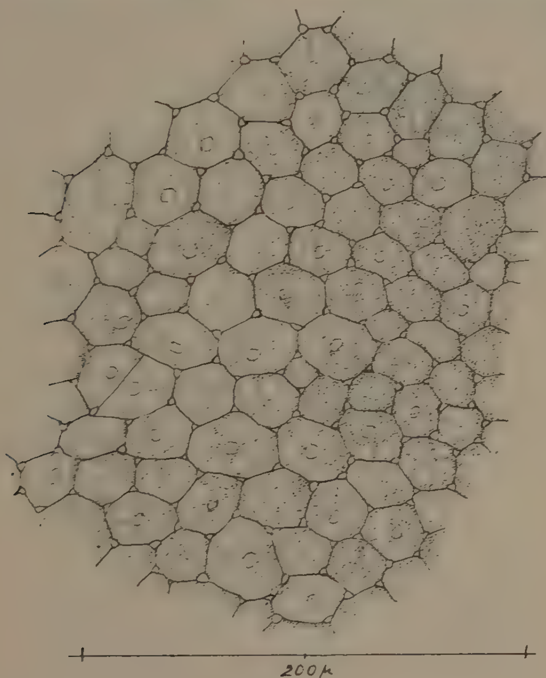

Fig. 6.

Transverse section through the meristem, about 0.05 mm higher than that of fig. 5; the intercellular spaces which are still triangular have sides which are rounded off outwards.

0.05 mm higher) shows that when the cells increase in size, the intercellular spaces soon change their form. They are still triangular, it is true, but the triangle sides now curve outwards. This may even result in a rounded shape. The latter is the case in a subsequent stage of development (see fig. 7) of most intercellular spaces.

Figures 8 and 9, representing transverse sections of still older stages, show that the round form again makes place for a triangular one in which, however, the verticles lie where at earlier stages were found the centres of the sides. In these stages the cells are already evidently stellate. We would however point out that the sides of the triangles in these stages are not perfectly straight, but a little indented in the middle. From our figure we may conclude further that the walls now begin to thicken.

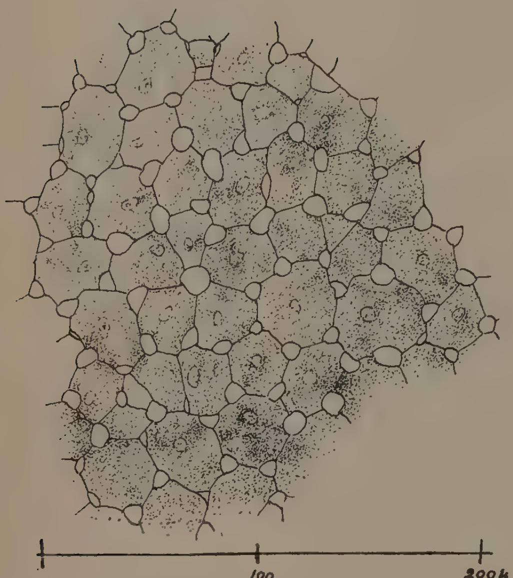

Fig. 7.

Transverse section through the meristem in a stage a little older than that of fig. 6; the intercellular spaces are rounded off.

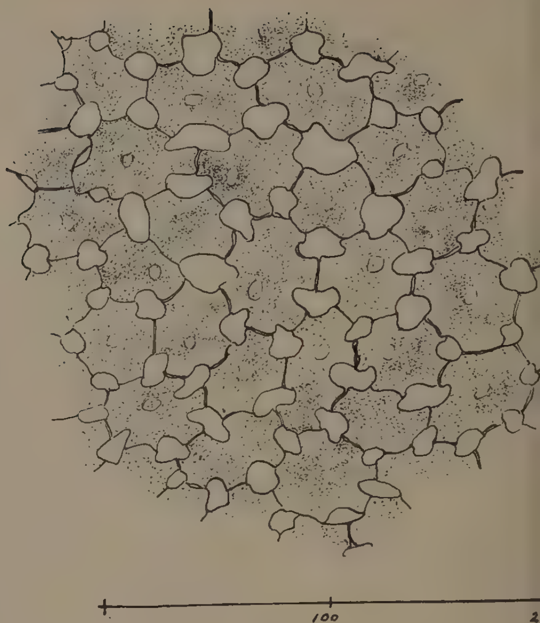

Fig. 8.

Transverse section through the meristem in a somewhat older stage than that of fig. 7; the cell arms become visible.

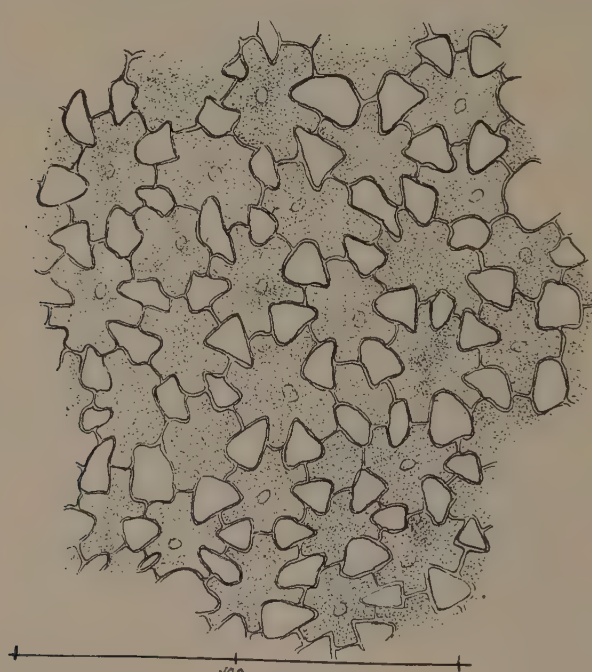

Fig. 9.

Transverse section through the meristem in a somewhat older stage than that of fig. 8; the cell arms are very evident.

When on transverse section the original meristem exclusively shows hexagons of the same size, all the pitch cells will show six arms on transverse section, if the development takes place as here described. Actually there also occur other sections than the hexagonal ones, and the cells are not all of the same size. So there are deviations in the number of arms and it also happens that cells have contact with each other through more than one pair of arms. Yet the average number of arms approaches the number six. LEWIS (12) calculated from the observation of 1000 cells an average of 5.6 arms per cell. On counting transverse sections of 157 cells we ourselves found 928 cell arms, and that with a frequency distribution as given in figure 10. From this an average value

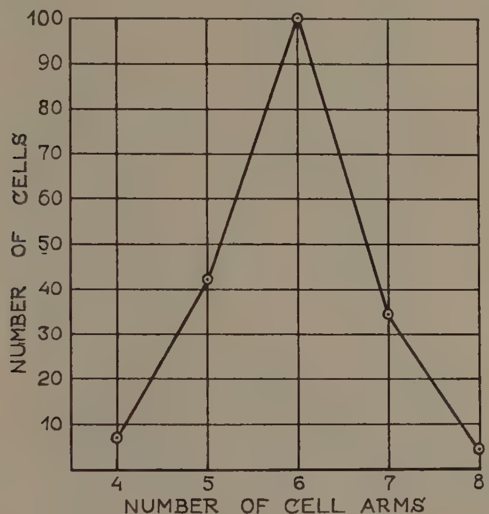

Fig. 10.

Frequency distribution of the number of cell arms in various pith cells of *Juncus effusus*.

of 5.9 ± 0.89 is calculated, in which the last number represents the mean error in the average value (found by applying the formula $f_m = \frac{\sigma}{\sqrt{n}}$ in which σ = the mean error of all observations and n = the total number of observations.

Figure 11 finally gives a picture of the final stage of the pitch on transverse section. The arms of the stellate cells have become much longer and thus the generally triangular intercellular spaces have become considerably larger. The sides of the triangles are now perfectly straight, owing to the walls of the arms being in line. The cell walls have again become thicker and have a diameter of 1 to 2 μ. The thickness of the walls, however, is not the same everywhere; the walls in the axil of two arms of the same cell are thickest. The short transverse walls at the end

of the arms, where they separate two arms from adjacent cells, are rather irregularly thickened and slightly corrugated.

Fig. 11.

Transverse section of the adult pith in the circular leaf of *Juncus effusus*; the cell walls of the stellate cells are thickened.

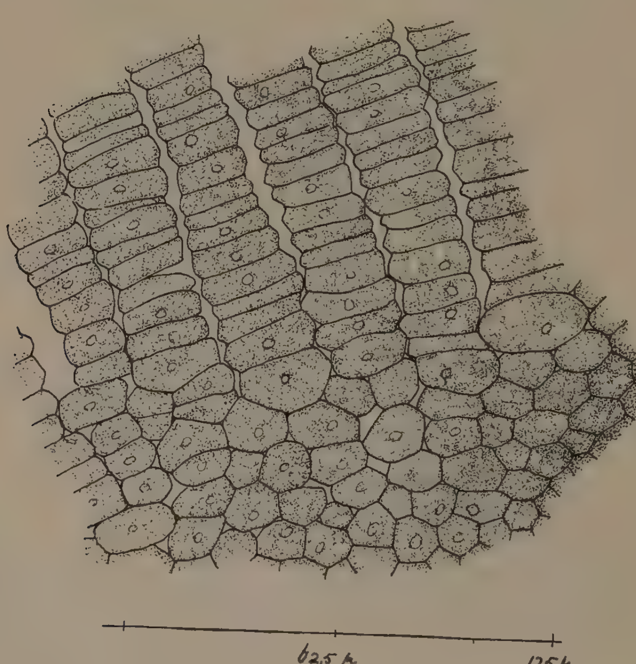

Fig. 12.

Longitudinal section of the meristem over the cleft-like hollow, bounding the growing point of *Juncus effusus*; at the bottom: polyhedral meristem with and without intercellular spaces; a little higher: formation of vertical rows of cells.

In this stage there is no longer much protoplasm observable, apparently the cells have died.

We shall now consider the development of the pith cells on vertical sections.

The lower part of figure 12 again shows that in the youngest meristem the cells are polygonal and placed together without any interstices. This meristematic zone is about 100 μ high. It is seen that above it there occur intercellular spaces and that higher still, without many transitional cells, vertical rows of cells become visible, which are separated by intercellular spaces which often seem to be continued over comparatively great distances.

However, the picture slightly differs on some sections from that shown in fig. 12; this may be seen in fig. 13 which gives a longitudinal section on a

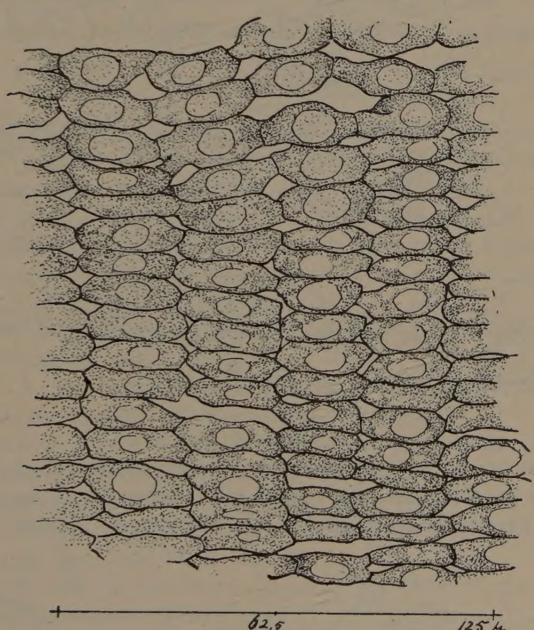

Fig. 13.

Longitudinal section of meristem of *Juncus effusus*, made in another place than that of fig. 11, with vertical cell rows; by the sides the cells are connected by cell arms which are as yet short — the circular figures on the cells are dissected cell arms —; at the top the cells begin to part more evidently.

somewhat higher level. Here too vertical rows of cells are visible, but the intercellular spaces situated between those rows are not continued in this preparation. On the other hand the coherence between the cells in the separate rows is broken in several places. Here the cells of neighbouring rows are in many places in contact with each other and we observe already the beginning of the formation of cell arms. We would draw the reader's attention to the fact that the circular figures seen in this and the two following figures represent sections of cell arms, projected on the cell body.

Fig. 14 shows a further stage of development of the pith in the vertical section. It approximately agrees with the stage of which fig. 8 gives a horizontal section. The cells in the vertical rows have now become entirely

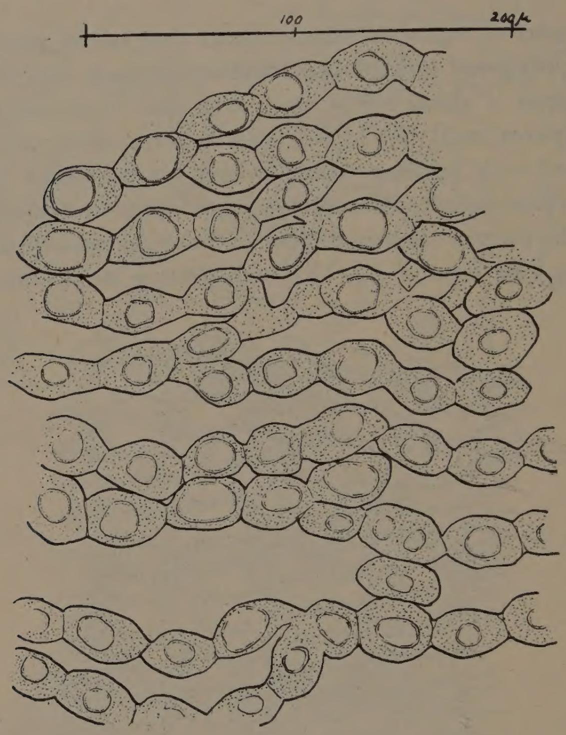

Fig. 14.
Longitudinal section of not quite adult pith in the circular leaf of *Juncus effusus;* most cell arms have attained some considerable size; at the top and at the bottom the cells have become practically detached without forming cell arms.

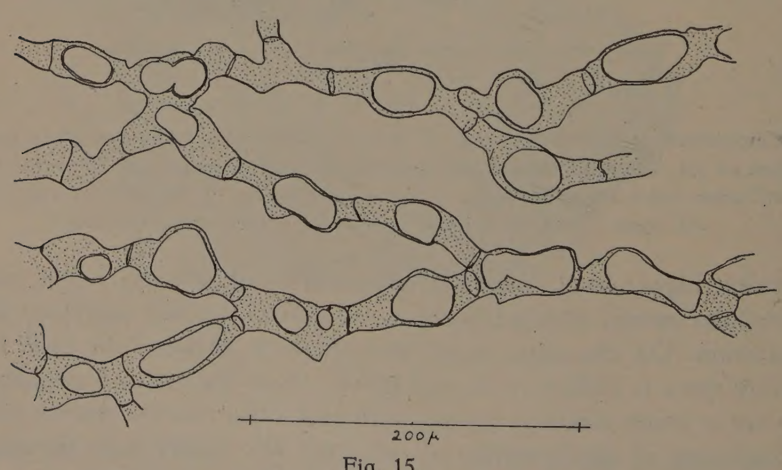

Fig. 15.
Longitudinal section of adult pith in the circular leaf of *Juncus effusus*; the cell arms have reached their ultimate length.

detached from each other, but those of neighbouring rows cohere through cell arms.

Fig. 15 gives the final stage in which the spaces between the attached cells are much enlarged.

We would finally also mention here that LEWIS has shown the plausibility of the assumption that the average cell shape of juvenile pith, for instance in a stage a little younger than that shown in fig. 13, is cubo-octahedral. The vertical cell rows are supposed to be accumulations of such bodies which are piled up on their hexagonal faces and the lateral walls of which are formed by 6 squares and 6 hexagonal faces. The 12 lateral faces are in contact with corresponding faces of adjacent rows of cells and from these faces 12 arms are supposed to develop, connecting these cells with the adjacent ones in the adult pith.

(To be concluded in the next number of these Proceedings.)

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